Mathematical Reviews

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HISTORY

***Archibald, Raymond Clare. Outline of the History of Mathematics.** 5th ed. Mathematical Association of America, Inc., Oberlin, Ohio, 1941. ii+76 pp. $.75.

***Ore, Oystein. Mathematics.** Development of the Sciences, Second Series, pp. 1–51. Yale University Press, New Haven, Conn., 1941.

This lecture attempts to give an outline of the development of mathematics from its beginnings up to the present time. For the decisive period of the 19th century it follows mainly the lines drawn in F. Klein's "Vorlesungen," but extends and completes this with respect to the preceding and following periods. There are a few instances where traditional but erroneous statements are repeated, but these are only small corrections to the interesting picture given by this report. *O. Neugebauer* (Providence, R. I.).

Langer, R. E. Alexandria—shrine of mathematics. Amer. Math. Monthly **48**, 109–125 (1941). [MF 3914]

***Neugebauer, Otto E. Exact science in antiquity.** University of Pennsylvania Bicentennial Conference, Studies in Civilization, pp. 23–31. University of Pennsylvania Press, Philadelphia, 1941.

Report about the development of mathematics and astronomy in antiquity. The main emphasis is laid on the influence of contact between heterogeneous cultures and the conditions which created the milieu in which astronomy and mathematics of the Hellenistic period originated.
O. Schmidt (Providence, R. I.).

***Schlesinger, Frank. Astronomy.** Development of the Sciences, Second Series, pp. 53–89. Yale University Press, New Haven, Conn., 1941.

This report on the development of astronomy is divided into two parts of different character. The larger first part (to page 78) gives the history of the investigations of our solar system up to Newton. The rest is devoted to the modern development giving, in wise selection from the enormous amount of material, such problems which can be considered characteristic of the astronomical research during the last two centuries. One must, however, raise several objections to the historical report in the first part, which might change the picture given by the author in essential points. If the author, for example, says that "eclipses were predicted . . . probably as early as 2000 B.C.," then we can only remark that it follows from the cuneiform material that such predictions were impossible even 1500 years later. On the other hand a question like the following "we cannot help wondering how far astronomy had progressed in Chaldea and whether the people who had discovered the saros may not have made other equally striking advances, the record of which has not come down to us" could easily have been answered by consulting Kugler's "Babylonische Mondrechnung," published in 1900 and giving the most detailed information about the Babylonian theory of the moon, not to mention numerous more recent publications during the past 40 years [the bibliography not only omits the works of Kugler, but also those of Delambre, Schiaparelli, Tannery, Heath, etc.]. Thales and Pythagoras are accredited with various discoveries, although these stories have long since been disproved, both by astronomers [e.g., Pannekoek, Nederl. Akad. Wetensch., Proc. **20**, 943 ff. (1917)] and historians [e.g., Frank, Plato und die sogenannten Pythagoreer [Halle, 1923], or Heidel, Amer. J. Philology **61**, 1 ff. (1940)]. No attempt is made to harmonize the history of ancient astronomy with our modern knowledge about the Hellenistic culture, its origins and development.

Many details could be rectified. I mention only that Hipparchus' discovery of the precession of the equinoxes is the least important of his contributions; this phenomenon did not become important until Newton's insight revealed that this secular displacement is one of the most striking tests for the law of general gravitation. The fact that the Chaldeans were not satisfied with a year length of $365\frac{1}{4}$ days is well known from numerous texts. The Almagest contains much more new material than is suggested by a sentence like "the larger portion of the Almagest is taken up with an account of Hipparchus' work"; I might only mention the improvements in the calculation of eclipses, the discovery and theoretical treatment of the evection, and the star catalogue [cf. Vogt, Astr. Nachr. **224**, no. 5354 (1925)]. Tycho Brahe's argument against the movement of the earth is based less on the enormous star distances, but mostly on the conclusion from it of star-diameters of the order of magnitude of the earth's orbit resulting from the further assumption of a finite visible diameter of the stars [opera II, 430], an assumption not disproved until the invention of the telescope. *O. Neugebauer* (Providence, R. I.).

***Ptolemy. Tetrabiblos.** Edited and translated into English by F. E. Robbins. Loeb Classical Library: William Heinemann, Ltd., London; Harvard University Press, Cambridge, Mass., 1940. xxiv+466 pp.

Almost simultaneously with the Teubner edition of Ptolemy's Tetrabiblos [cf. these Rev. **2**, 4] there appeared a new edition in the Loeb collection, which is based on an independent investigation of a certain group of the manuscripts (representing, however, all families). The differences resulting are not very large, but the numbering of the chapters is sometimes different. The English translation is accompanied by commentaries in the notes.
O. Neugebauer (Providence, R. I.).

Chatley, Herbert. Egyptian astronomy. J. Egyptian Archaeol. **24**, 120–126 (1940). [MF 4878]

Report mainly on the diagonal calendars.
O. Neugebauer (Providence, R. I.).

van der Waerden, B. L. Zur babylonischen Planetenrechnung. Eudemus 1, 23–48 (1941). [MF 4720]

This paper is mainly concerned with the Babylonian theory of the movement of Jupiter, but also considers Saturn and Mars. The main result is the establishment of the parameters of the kinematic "model" for each of the three different methods for calculating the movement of Jupiter. The empirical elements of the theory are discussed and the relations connecting the calculation of the longitudes and the calculation of the dates of the characteristic phenomena are explained. Of special interest is the discovery that the most highly developed system even takes the anomaly of the movement of the sun into consideration. The introduction of artificial "days" of length 1/30 of the average length of the synodic month is explained as a means for avoiding the irregularities of time measurement with respect to the equatorial system. The longitude of the Babylonian origin of the ecliptic is found to be about $-4°$.

O. Neugebauer (Providence, R. I.).

✱Neugebauer, Otto E. Some fundamental concepts in ancient astronomy. University of Pennsylvania Bicentennial Conference, Studies in the History of Science, pp. 13–29. University of Pennsylvania Press, Philadelphia, 1941.

This paper is devoted to a study of methods in ancient astronomy developed from the problem of determining the variable length of day and night. This leads to the problem of investigating the "oblique-ascension" or "rising-times" of arcs of the ecliptic. The present paper reports about "linear methods" developed in Babylonian and Greek astronomy for finding approximate solutions for these problems. *O. Schmidt* (Providence, R. I.).

Luckey, P. Zur Entstehung der Kugeldreiecksrechnung. Deutsche Math. 5, 405–446 (1941). [MF 4253]

This paper is mainly concerned with the sine theorem of Arabian spherical trigonometry, which replaced the theorem of Menelaos which was the main instrument of Greek trigonometry. An introductory chapter gives a careful discussion of its history, using unpublished material in various instances. There follows the translation of a new text, a letter of Abū Naṣr to al-Bīrūnī, of which the only manuscript is now at Bankipore, India. This letter contains two proofs of the sine theorem and shows its applications to classical problems of spherical astronomy. The paper ends with commentaries on this text and a systematic comparison between the methods of Ptolemy and Abū Naṣr. *O. Neugebauer.*

Gupta, Hansraj. On the extraction of square-root of surds. Proc. Benares Math. Soc. (N.S.) 2, 33–37 (1940). [MF 4352]

"This note should be read as an addendum to Dr. A. N. Singh's paper 'On the arithmetic of surds among the ancient Hindus,' Mathematica 12, 102–115 (1936)."

Author's note.

Fujiwara, M. Miscellaneous notes on the history of chinese mathematics. III. Mathematics in the old Korea. Tôhoku Math. J. 47, 309–321 (1940). (Japanese) [MF 4031]

Fujiwara, Matsusaburô. A brief sketch of the Wazan, the mathematics of the old Japanese school. J. Sendai Cultural Soc. 1941, 64–84 (1941). [MF 4072]

Fujiwara, M. Miscellaneous notes on the history of Wazan. VI. Newton's interpolation formula in Wazan. Tôhoku Math. J. 47, 322–338 (1940). (Japanese) [MF 4032]

Katô, Heizaemon. On the catenary in the old Japanese mathematics. Tôhoku Math. J. 47, 279–293 (1940). (Japanese) [MF 4028]

Krishnaswami Ayyangar, A. A. Rene Descartes. Math. Student 8, 101–108 (1940). [MF 4311]

Zacharias, Max. Desargues' Bedeutung für die projektive Geometrie. Deutsche Math. 5, 446–457 (1941). [MF 4254]

Summary of the content of Desargues' "Brouillon." The relations to Apollonius and Pappus are pointed out, and various errors in modern books are corrected.

O. Neugebauer (Providence, R. I.).

Hofmann, Josepha und Hofmann, Jos. E. Erste Quadratur der Kissoide. Deutsche Math. 5, 571–584 (1941). [MF 4805]

Sergescu, P. Mathématiciens français du temps de la Révolution Française. An. Acad. Romane. Mem. Sect. Ştiinţifice (3) 16, no. 2, 48 pp. (1940). [MF 4125]

Steck, Max. Ein unbekannter Brief von Gottlob Frege über Hilberts erste Vorlesung über die Grundlagen der Geometrie. S.-B. Heidelberger Akad. Wiss. 1940, no. 6, 8 pp. (1940). [MF 4124]

Blaschke, Wilhelm. Obituary: Hermann Brunn. Jber. Deutsch. Math. Verein. 50, 163–166 (1940).

Huber, A. Obituary: Philipp Furtwängler. Jber. Deutsch. Math. Verein. 50, 167–178 (1940).

Kolmogoroff, A. Obituary: Valerii Ivanovich Glivenko. (1897–1940). Uspekhi Matem. Nauk 8, 379–383 (1941). (Russian) (1 plate) [MF 4946]

Lusternick, L. Obituary: Dmitrii Aleksandrovich Grave. (1863–1939). Uspekhi Matem. Nauk 8, 377–378 (1941). (Russian) (1 plate) [MF 4945]

Obituary: Dmitrii Aleksandrovič Gravé. 1863–1939. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 349–356 (1940). (Russian) [MF 4078]

Obituary: Ivan Ivanovič Ivanoff. 1862–1939. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 357–362 (1940). (Russian) [MF 4079]

Carathéodory, Constantin. Obituary: Ferdinand von Lindemann. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1940, 61–63 (1940). [MF 4575]

Thompson, D'Arcy W. and Chapman, S. Obituary: Prof. Vito Volterra, For. Mem. R. S. Nature 147, 349–350 (1941). [MF 4201]

Levi, Beppo. The personality of Vito Volterra. Publ. Inst. Mat. Univ. Nac. Litoral 3, 25–36; list of publications, 37–48 (1941). (Spanish) [MF 4953]

Wilson, G. H. A. Obituary: Dr. Alfred Young, F.R.S. Nature 147, 229 (1941). [MF 4138]

THEORY OF GROUPS

Miller, G. A. Maximal subgroups of a finite group. Proc. Nat. Acad. Sci. U. S. A. **27**, 212–216 (1941).

A group G_k which has exactly k maximal proper subgroups can be constructed for each $k\geqq 1$ by forming the direct product of k composite cyclic prime power groups with different primes. For $k=1, 2$ there are no other G_k's than these. For $k=3$, the only other G_3's are groups of order 2^m having two independent generators. The case $k=4$ is discussed completely by the author, and several general theorems about maximal proper subgroups of a group are proved. *J. S. Frame* (Providence, R. I.).

Cheissin, G. Die Klassifikation von Gruppen, deren Ordnung p^2q^2 ist. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 535–551 (1940). (Russian. German summary) [MF 4091]

In this paper it is assumed that $p>q$ and $p^2q^2\neq 36$. With these restrictions there are four sets of groups according to whether the subgroups of orders p^2 and q^2 are cyclic or not. The method of classification is based on O. Schreier's method of "extension" of groups [O. Schreier, Abh. Math. Sem. Hansischen Univ. **4** (1926)]. The number of distinct-non-isomorphic-types in each set depends on the solutions of congruences of the form $x^{q^2}\equiv 1 \pmod{p^2}$ and the structure of each type is given. *M. S. Knebelman.*

Frame, J. S. The double cosets of a finite group. Bull. Amer. Math. Soc. **47**, 458–467 (1941). [MF 4538]

The number of self-inverse double cosets of a finite group G with respect to a subgroup H is equal to the sum of the multiplicities μ_i^H of those irreducible components Γ_i of G appearing in the permutation representation G_H of G which have a symmetric bilinear invariant minus the sum of the multiplicities of those which have an alternating invariant. The proof of this theorem is based upon a paper of Frobenius and Schur [S.-B. Berlin Math. Ges. **1906**], and the result is the generalization of a corresponding theorem of the author's for the symmetric group. Let the degree of G_H be $n(=g/h)$ and the degree of Γ_i be n_i. If the operations of H permute the n symbols in r transitive sets of k_i symbols each, let $K=\Pi_{i=1}^r(k_i)$ and $N=\Pi_{i=1}^r n_i^{(\mu_i^H)^2}$; then the author also proves that $n^{r-1}K/N=\bar{P}_1P_1$, where P_1 is an algebraic integer in the field of the characters of the components of G_H. *G. de B. Robinson* (Toronto, Ont.).

Thrall, Robert M. A note on a theorem by Witt. Bull. Amer. Math. Soc. **47**, 303–308 (1941). [MF 4182]

Let F denote the free group with n generators, let H_k be the subgroup generated by the kth powers in F. Set $F_k=F/H_k$, and set $G_{k,c}=F_k/F_k^{c+1}$, where F_k^{c+1} is the $(c+1)$st member of the lower central series of F_k. The group $G_{k,c}$ may be called the free k-group of class c. Using a result of Witt [J. Reine Angew. Math. **177**, 152–160 (1937)], Thrall determines the order of the factor groups F_k^c/F_k^{c+1} provided that k is a power of a prime p and that $c<p$. Further, the characteristic subgroups of $G_{p,c}$, $c<p$, are studied. Let M_c denote the space of tensors of rank c over the finite field $GF(p)$. A homomorphic mapping of M_c, taken as an additive group, upon the central C of $G_{p,c}$ is set up. Using the known decomposition of these tensors into irreducible constituents, the minimal characteristic subgroups of the free k-group of class c can be obtained. Any characteristic subgroup of $G_{p,c}$ which lies in the center is a direct product of minimal characteristic subgroups. *R. Brauer* (Toronto, Ont.).

Fuchs-Rabinowitsch, D. I. Beispiel einer diskreten Gruppe mit endlichvielen Erzeugenden und Relationen, die kein vollständiges System der linearen Darstellungen zulässt. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 549–550 (1940). [MF 3969]

The author points out that a certain element, not 1, in a group discussed in a former paper of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 425–426 (1940); cf. these Rev. **2**, 126] is mapped upon the unit matrix in every representation of the group by matrices.
R. Baer (Urbana, Ill.).

Dickman, A. P. Some theorems on infinite groups. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 63–67. (Russian) [MF 3503]

Let G be a group, $(a_1, a_2, \cdots, a_k)$ a finite class of conjugate elements of G. Let β be the order of a_1. The author proves that each element of the group H generated by the elements a_i can be represented in the form $a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_k^{\alpha_k}$ with $0\leqq\alpha_i<\beta$ $(i=1, 2, \cdots, k)$. This implies that H is a finite group provided the a_i are of finite order. Easy consequences of this proposition are several known results concerning finiteness of decompositions of groups in cosets with respect to simple or double modules [see A. P. Dickman, C. R. (Doklady) Acad. Sci. URSS (N.S.) **15**, 71–76 (1937); W. Turkin and P. Dubuque, C. R. Acad. Sci. Paris **205**, 435–437 (1937)]. Using the above-mentioned theorem the author proves further that a p-group (that is, a group in which orders of all elements are powers of the prime number p) which contains a finite class different from the identity has a non-vanishing center. [This result was first established by the author in his paper quoted above.]
W. Hurewicz (Chapel Hill, N. C.).

Wedderburn, J. H. M. Homomorphism of groups. Ann. of Math. (2) **42**, 486–487 (1941). [MF 4297]

Homomorphisms are considered here as many to many correspondences. Let two groups be divided into mutually exclusive sets, $G=G_1+G_2+\cdots$, and $H=H_1+H_2+\cdots$, and write g_i for a typical element of G_i. The author then calls G homomorphic to H $(G\sim H)$ if, under the many to many correspondence $G_p\rightleftarrows H_p$, $g_pg_q=g_r$ implies $H_pH_q=H_r$. The main result is: Theorem: If $G\sim H$, then also $H\sim G$. If G_1 contains the identity of G, then $H_1\sim G_1$ contains the identity of H. Further G_1 and H_1 are invariant subgroups of G and H, respectively, and $G/G_1\cong H/H_1$. *M. Hall.*

Levi, F. W. The commutatorgroup of a free product. J. Indian Math. Soc. (N.S.) **4**, 136–144 (1940). [MF 4281]

If A_i' is the commutator subgroup of the group A_i, and if the elements $a(i, v)$ are representatives of the cosets of $A_i/A_i'(a(i, 0)=1)$, then the author proves that the commutator subgroup of the free product of A_1 and A_2 is exactly the free product of the groups $a(i, v)A'_{i+1}a(i, v)^{-1}$ and of the infinite cyclic groups generated by the elements $a(1, v)a(2, w)a(1, v)^{-1}a(2, w)^{-1}$ (for v and w not 0).
R. Baer (Urbana, Ill.).

Turri, T. Sul gruppo delle omografie che trasformano in sè una correlazione. Rend. Sem. Fac. Sci. Univ. Cagliari 10, 69–87 (1940). [MF 4230]

The author starts with a survey of the known facts about the groups of collineations which transform a given correlation in itself, referring chiefly to E. Bertini [Geometria proiettiva degli iperspazi, 1923], where the theory of elementary divisors and their geometrical representation as developed by Segre and Pedrella is extensively used. He then investigates the composition of these groups and the number of parameters on which they depend. A. Voss has given methods for computing these numbers, but they become complicated if the given correlation is not irreducible. The author shows how to overcome these difficulties, using in a systematical way the collineation which is the square of the given correlation, and supposing that the elementary divisors of the characteristic matrix of this collineation are known. *E. Helly* (Paterson, N. J.).

Iwasawa, Kenkiti. Über die Einfachheit der speziellen projektiven Gruppen. Proc. Imp. Acad. Tokyo 17, 57–59 (1941). [MF 4452]

If K is a commutative field, n an integer exceeding 1, then $SL(n, K)$ denotes the group of all n-rowed square matrices with coefficients in K whose determinant is 1, and $PSL(n, K)$ is the group of those transformations of $(n-1)$-dimensional projective space that may be represented by matrices in $SL(n, K)$, so that $PSL(n, K)$ is essentially the same as the quotient-group of $SL(n, K)$ modulo its centrum. The author shows the equivalence of the following three statements: (1) $PSL(n, K)$ is simple. (2) $SL(n, K)$ equals its commutator subgroup. (3) Either $2<n$ or K is neither the prime field of characteristic 2 nor the prime field of characteristic 3. *R. Baer* (Urbana, Ill.).

Montgomery, Deane. Remarks on groups of homeomorphisms. Amer. Math. Monthly 48, 310–317 (1941). [MF 4515]

This paper, presented at the Baton Rouge meeting of the Mathematical Association of America in January, 1941, is a discussion without proofs of work of Brouwer, Kerékjártó, Newman, P. A. Smith, von Neumann, Montgomery and Zippin, M. Richardson and Eilenberg. It includes a full list of references to serve as introduction to the field of groups of homeomorphisms, and explains the more fundamental ideas involved, such as metric space, homeomorphism, equivalence of homeomorphisms, group of homeomorphisms and topological group. Many of the known results are given as a setting for the description of some of the outstanding problems in the field. *W. W. Flexner* (Ithaca, N. Y.).

Krull, Wolfgang. Über separable, abgeschlossene Abelsche Gruppen. J. Reine Angew. Math. 182, 235–241 (1940). [MF 3439]

The groups considered in this paper are primary in the sense that they admit the ring of p-adic integers as operators. A primary Abelian group A is of rank one if it has one of the types: (a) the additive group $\mathfrak{R}$ of p-adic numbers; (b) the additive group $\mathfrak{K}$ of all p-adic numbers; (c) $\mathfrak{R}/p^n\mathfrak{R}$; (d) $\mathfrak{K}/\mathfrak{R}$. The group A is separable and closed if it contains a sequence of subgroups $A=A_0>A_1>A_2>\cdots$ such that (1) the intersection $\Delta A_i=0$; (2) A_i/A_{i+1} have type (c) or (d); (3) if α_i is a sequence of elements of A such that $\alpha_{i+1}\equiv\alpha_i(A_i)$, then $\xi\equiv\alpha_i(A_i)$ is solvable in A. Let E be the closed subgroup of A generated by the elements of finite order. Then A is said to be correctly knotted if (1) A/E has no elements of finite order and (2) the subgroup of elements of infinite height of E is generated by elements of finite order. The main result of this paper is that A is a direct sum of groups of rank one if and only if it is correctly knotted. The results have applications to the theory of infinite Abelian extensions of a field. *N. Jacobson* (Chapel Hill, N. C.).

Kulikoff, L. Zur Theorie der Abelschen Gruppen von beliebiger Mächtigkeit. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 165–181 (1941). (Russian. German summary) [MF 4494]

The paper deals mainly with primary Abelian groups and extends the work of H. Prüfer and R. Baer. Among many interesting results obtained we quote Theorem 2: A primary Abelian group H contains no element of infinite height when and only when every finite sequence of elements of H can be included in a finite serviceable subgroup of H. Theorem 3: In order that a denumerable Abelian group be decomposable into a direct sum of cyclic groups it is necessary and sufficient that every finite sequence of elements may be included in a serviceable subgroup with a finite number of generators. From these theorems follows Pontrjagin's criterion for decomposability of torsionless Abelian groups into direct sums of cyclic groups and also Prüfer's criterion for denumerable primary Abelian groups (without elements of infinite height). The last theorem of the paper establishes the existence of primary groups of arbitrary power $\aleph_{\alpha+1}$ not containing any elements of infinite height and not decomposable into a direct sum of groups of lower power. *M. S. Knebelman.*

Toyoda, Kôshichi. On linear functions of abelian groups. Proc. Imp. Acad. Tokyo 16, 524–528 (1940). [MF 4327]

Continuing his work [Tôhoku Math. J. 46, 239–251 (1940); these Rev. 2, 6 (referred to below as A); Proc. Imp. Acad. Tokyo 16, 161–164 (1940), these Rev. 2, 6] on a set G, axiomatically defined, which generalizes the arithmetic mean, the author defines G by means of an operation which associates with n elements $a_1, \cdots, a_n$ of G an element $(a_1, \cdots, a_n)$ of G subject to an associative law, existence of the identity: $(0, \cdots, 0)=0$, and unique solvability on the first two coordinates: $(x, y, 0, \cdots, 0)=a$ implying x (or y) uniquely determined by y (or x). After proving two lemmas at considerable length and showing that the convention $x=(a, 0, \cdots, 0)$, $y=(0, b, 0, \cdots, 0)$, $x+y=(a, b, 0, \cdots, 0)$ makes G an abelian additive group, he states his main theorems without either discussion or proof. These are: (1) $(x_1, \cdots, x_n)=A_1x_1+\cdots+A_nx_n$, where $A_iA_j=A_jA_i$, and A_i is an automorphism (homomorphism) of G for $i\leqq 2$ $(i>2)$; (2) if the elements of G are real numbers and $(x_1, \cdots, x_n)$ is continuous in each variable, there is a 1-1 bicontinuous f such that $f(x_1, \cdots, x_n)=\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)$, λ_i real numbers. The author concludes by introducing another axiomatic system which turns out to be similar to that of A except that the automorphisms Γ_1, Γ_2 [see these Rev. 2, 6] do not necessarily commute. *W. W. Flexner.*

Samelson, Hans. Über die Sphären, die als Gruppenräume auftreten. Comment. Math. Helv. 13, 144–155 (1940). [MF 4442]

This paper essentially contains a topological proof for the following result: A compact Lie group of rank one, that is, in which maximal abelian subgroups are of dimension 1, is homeomorphic to S_1 (the circle), S_3 (the 3-sphere) or P_3 (projective 3-space). Combining this with a special case of H. Hopf's results [cf. the preceding review], a simplified

proof for the relevant part of which is given is an appendix, it is deduced that the only group-spaces which have the same Betti numbers as a sphere are S_1, S_3 and P_3. All this, of course, was previously known from the general theory of semi-simple groups.

G being a compact group, L a maximal abelian subgroup and L being assumed to be one-dimensional, the normalizer N of L in G is introduced. Either $N=L$, or L is of index 2 in N [this follows best from H. Hopf, loc. cit., no. 23, and from the fact that the group of automorphisms of L is of order 2]. The author considers the homogeneous spaces G/N and G/L (spaces of cosets in G modulo N and modulo L); G/N is the same as the space of all subgroups conjugate to L, that is, in the present case of all one-parameter subgroups of G (this, because both are manifolds of the same dimension, one of which is homeomorphically mapped into the other; or from A. Weil [C. R. Acad. Sci. Paris **200**, 518–520 (1935)], this is a projective space, as follows from the consideration of infinitesimal transformations in G). G/L is either the same as G/N (if $N=L$), or a two-sheeted covering of G/N and therefore a sphere; a more detailed discussion shows that (barring the trivial case $G=L$) G/L is a sphere of dimension not less than 2, and that $2L$, considered as a cycle on G, is ~ 0 (homologies are taken with the ring of integers as coefficients). So it remains to prove the following: If M is a fibre-space, fibres F being circles, over a sphere of dimension not less than 2, and if $2F\sim 0$ in M, then $M=S_3$ or $M=P_3$. The proof uses Feldbau's theorem [C. R. Acad. Sci. Paris **208**, 1621–1623 (1939)] and an "addition-theorem," the base-space being represented as the union of two hemispheres and the fibre-space being decomposed accordingly. *A. Weil* (Haverford, Pa.).

Nisnewitsch, V. L. Über Gruppen, die durch Matrizen über einem kommutativen Feld isomorph darstellbar sind. Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 395–403 (1940). (Russian. German summary) [MF 3738]

In this paper the author examines the conditions under which a group may have a true (isomorphic) representation in matrices with elements in a commutative field Γ. It is shown that such a representation exists for a direct product G of groups G_α if and only if there exists one such for each G_α. Should each G_α have a true representation in matrices of order n, then G would have a true representation in matrices of order $n+1$. The author considers next a group G with a denumerable number of elements and proves that it possesses a representation in matrices of order n over a commutative field, should every finite subgroup of G possess such a representation. In carrying out the desired constructions he introduces matrices X_α whose elements are independent variables $x_{ik\alpha}$ and obtains the required representations in matrices with elements in the quotient fields of certain subrings of $\Gamma[x_{pq1}, \cdots, x_{rs\alpha}, \cdots]$. The author states that A. I. Malcev established the truth of the second theorem for groups with nondenumerable number of elements. The last section of the paper contains an example of a group with two generators which cannot be characterized by a finite number of relations between these generators.

A. E. Ross (St. Louis, Mo.).

Turkin, W. K. On characters of monomial groups. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 259–264. (Russian) [MF 3518]

Employing the ideas of an earlier paper [Math. Ann. 111, 743–747 (1935)] and a property (Th. I) of the characters of a transitive monomial representation of a finite group G, the author obtains the following criteria for the existence of invariant subgroups of G. Let H be a subgroup of G of index f and $G=H+Hg_2+\cdots+Hg_f$. Let A be an element of G and let the class of conjugate elements containing A have h members. Then (Th. II) if $(h, f)=1$, either the character $\chi(A)=0$ or G has an invariant subgroup. If A is in H and its order is p^λ, where p is a prime not dividing f, and if no power A^x $(x\neq p^\lambda)$ is in the commutator of H, then (Th. IV), should A belong to fewer than $2f/h$ groups among H, $g_1^{-1}Hg_1, \cdots, g_f^{-1}Hg_f$, the group G would have an invariant subgroup. *A. E. Ross* (St. Louis, Mo.).

Brauer, R. and Nesbitt, C. On the modular characters of groups. Ann. of Math. (2) **42**, 556–590 (1941). [MF 4302]

The authors first summarize the theory of modular characters of a finite group $\mathfrak{G}$ as previously developed by them [Univ. of Toronto Studies, Math. Series, no. 4, 1937]. If an element of $\mathfrak{G}$ is defined to be p-regular when its order is prime to p, one of the important results in this theory is that the number of distinct absolutely irreducible modular representations of $\mathfrak{G}$ is equal to the number of classes of conjugate p-regular elements of $\mathfrak{G}$. The principal problem considered in this paper is the reduction of an ordinary representation of $\mathfrak{G}$ when the coefficients are taken in a field of characteristic p. It is proved that, if $g=p^a g'$, where $(g', p)=1$ and g is the order of $\mathfrak{G}$, then an ordinary irreducible representation Z_i of degree $z_i\equiv 0 \pmod{p^a}$ remains irreducible as a modular representation and the character $\zeta^{(i)}$ of Z_i vanishes for all elements of an order divisible by p. The theory of "blocks," which is essential to the argument, cannot be explained here. As in the ordinary theory, the Kronecker product of two modular representations of $\mathfrak{G}$ yields a new modular representation whose reduction is discussed. It is shown that the number of self-contragredient modular characters is equal to the number of self-reciprocal p-regular classes of $\mathfrak{G}$. Again, there is a modular theory corresponding to the ordinary theory relating the representations of $\mathfrak{G}$ to those of a subgroup $\mathfrak{H}$. As would be expected, the analogy is very close. The paper concludes with a number of special cases and applications of the theory to $GLH(2, p^a)$, $SLH(2, p^a)$ and $LF(2, p^a)$.

G. de B. Robinson (Toronto, Ont.).

Kawada, Yukiyosi. Über die Überlagerungsgruppe und die stetige projektive Darstellung topologischer Gruppen. Jap. J. Math. **17**, 139–164 (1940). [MF 4283]

Let $\mathfrak{G}$ be a topological group. A projective representation of $\mathfrak{G}$ of order n is a homomorphic mapping of $\mathfrak{G}$ into the quotient group of the full matric group of order n modulo its central. The present study concerns the topology of such representations. It is shown that a given projective representation of $\mathfrak{G}$ of order n can be induced in a certain way by an ordinary (continuous) representation of an at most n-fold covering group of $\mathfrak{G}$. A consideration of the totality of projective representations of $\mathfrak{G}$ leads to the definition of the multiplicator (Schur), a discrete group $\mathfrak{M}$ associated with $\mathfrak{G}$. For compact metric connected locally-connected groups it is shown that $\mathfrak{M}(\mathfrak{G}\times\mathfrak{G}')=\mathfrak{M}(\mathfrak{G})\times\mathfrak{M}(\mathfrak{G}')$, and that $\mathfrak{M}(\mathfrak{G})=1$ if $\mathfrak{G}$ is abelian. (These relations are not in general true for finite $\mathfrak{G}$'s.) If it is further assumed that $\mathfrak{G}$ is locally simply-connected, then $\mathfrak{M}$ can be explicitly defined in terms of the characters of a certain discrete subgroup of the universal covering group of $\mathfrak{G}$. It is a corollary

that the multiplicator of a compact semi-simple Lie group is isomorphic to its fundamental group. Numerous results are stated and proved for groups which are not necessarily simply connected locally. The necessary preliminary treatment of covering groups, fundamental groups, and so on for these more general topological groups is given in part I.
P. A. Smith (New York, N. Y.).

Vandiver, H. S. The elements of a theory of abstract discrete semi-groups. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 71–86 (1940). [MF 4406]

A semi-group is a set of elements for which an associative product is defined. If there is an identity in S then the author calls it a gruppoid. If $ab=ac$ and $ba=ca$ each imply $b=c$ (cancellation law) the author calls S a quasi-group. If S has an element r such that $rb=rc$ and $br=cr$ each imply that $b=c$, then S has an identity. The principal result is the following. Let S be finite and have a subgroup G of order n, the identity of which is an identity for all of S. Let s_i be the number of distinct elements in the coset C_iG. Then the number of elements in S is $\sum n/s_i$, summed over all cosets. Curiously enough, cosets are so defined in the general case that the coset to which an element is assigned depends not only on the choice of its representative C_i, but upon the order in which the representatives are selected as well. Some applications to number theory are given.
H. H. Campaigne (Minneapolis, Minn.).

Ore, Oystein. Remarks on structures and group relations. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 1–4 (1940). [MF 4397]

The author determines a set of necessary and sufficient conditions upon three elements of an arbitrary structure in order that they shall generate a Dedekind substructure, and in order that they shall generate a distributive substructure. As applications, he proves that, if A, B and C are subgroups of a given group such that each of them is permutable with the two others and with their cross-cut, then the structure generated by A, B and C is a Dedekind structure. In particular, two normal subgroups A, B and an arbitrary third subgroup C generate a Dedekind structure; and if, in addition, $C\cap(A\cup B)=(C\cap A)\cup(C\cap B)$, then A, B and C generate a distributive structure.
H. S. Wall.

ANALYSIS

Functional Equations and Operational Calculus

Hornstein, M. Équations régulières linéaires en différences finies. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 592–596 (1941). [MF 4467]

The background to this work is given by certain developments of H. Poincaré and O. Perron regarding linear difference equations (1) $D_{\nu+n}+a_1^{(\nu)}D_{\nu+n-1}+\cdots+a_n^{(\nu)}D_\nu=0$ ($\nu=0, 1, \cdots$). If $\lim a_i^{(\nu)}=a_i$ (as $\nu\rightarrow\infty$) exist, (1) is said to be of Poincaré type; of importance then is the characteristic equation $p^n+a_1p^{n-1}+\cdots+a_n=0$, with roots $p_1, \cdots, p_n$. A solution D_ν is regular if $\lim D_{\nu+1}/D_\nu=p$; it then belongs to p; (1) is said to be regular if all of its solutions are regular. Hornstein shows that there exist regular equations (1) not of Poincaré type. It is also proved that, if (1) is regular, all of its solutions can be grouped into classes $1, 2, \cdots, n$, so that the members of class k ($1\leqq k\leqq n$) belong to a number p_k (fixed for the class); letting $D_\nu^{(k)}$ be in class k, one has $\lim (D_\nu^{(s)}/D_\nu^{(r)})=0$ (for $s>r$); also, linear relations are given for the initial conditions for the solutions of class k.
W. J. Trjitzinsky (Urbana, Ill.).

Birkhoff, George D. and Guenther, Paul E. Note on a canonical form for the linear q-difference system. Proc. Nat. Acad. Sci. U. S. A. **27**, 218–222 (1941).

This note is concerned with the system of linear q-difference equations

$$y_i(qx)=\sum_{j=1}^{n} a_{i,j}(x)y_j(x), \qquad i=1, 2, \cdots, n,$$

in which $|q|\neq1$, the coefficients $a_{i,j}(x)$ are analytic or have poles at $x=\infty$, and the determinant $|a_{i,j}(x)|$ is not identically zero. It announces for this system the existence of a canonical form

$$y_i(qx)=\sum_{j=1}^{i} x^{\nu_j}q^{\rho_j}c_{i,j}(x)y_j(x),$$

in which $c_{i,i}=1$, and each $c_{i,j}(x)$ with $j<i$ is a polynomial in $1/x$. The method of deduction of this form is outlined, and detailed treatment is promised in later papers.
R. E. Langer (Madison, Wis.).

Arany, Daniel. Intégration de deux équations aux différences finies linéaires à deux variables. Acta Univ. Szeged. Sect. Sci. Math. **10**, 42–47 (1941). [MF 4430]

The paper deals with the difference equation

$$y(x, t)=py(x-1, t+1)+qy(x-1, t-1)$$

with $p+q=1$, and with either of the following sets of boundary conditions: (i) $y(x, 0)=0$, $y(x, n)=0$, $y(x, x)=q^x$, $y(x, t)=0$ if $x<t$; (ii) $y(x, 0)=0$, $y(x, n)=0$, $y(1, \alpha-1)=p$, $y(1, \alpha+1)=q$ with $1<\alpha<n-1$. A solution for each case is obtained, and an application to the theory of probability is mentioned.
R. E. Langer (Madison, Wis.).

Heins, Albert E. On the solution of partial difference equations. Amer. J. Math. **63**, 435–442 (1941). [MF 4163]

Consider the difference equation

$$(1) \qquad F(x+1, t)+F(x-1, t)=2F(x, t+1),$$

subject to the condition that $F(x, t)$ is prescribed in a strip of unit width bordering an infinite strip, above the x axis, of width $p+2$, p an integer. The Laplace transformation is applied for the variable t and yields a nonhomogeneous difference equation in x and a parameter s. The solution (2) of the difference equation subject to the transformed boundary conditions is found and the Mellin transform of (2) considered as a function of s yields a solution of (1). For $1\leqq x\leqq p$ (2) may be expanded in a finite number of partial fractions and the Mellin transform may be written down at once. This type of procedure is familiar in differential equations. Questions of uniqueness and existence conditions for the solutions of the general case associated with (1) are not treated.

The author's introduction of Nörlund sums for (2) is superfluous for, under the restrictions imposed, merely the usual successive approximations solution is obtained. Moreover, since only finite sums appear in the end result, the conditions of convergence of certain intermediate infinite series and integrals may be avoided [indeed, it is not shown that these convergence conditions are satisfied for the key function $\psi(x, s)$ introduced in the paper].
D. G. Bourgin (Urbana, Ill.).

Meredith, C. A. On a non-linear difference equation in two variables. J. London Math. Soc. 15, 260–272 (1940). [MF 4037]

V. A. Bailey proposed, as an approximation to the interaction of hosts and parasites, the non-linear system of difference equations $h_{n+1}=Eh_ne^{-ap_n}$, $p_{n+1}=Eh_n-h_{n+1}$, where h_n, p_n are the populations in the nth generation, $E>1$ is the fertility of the hosts and a is a positive constant. The equilibrium point is

$$h_n=(\log E)/a(E-1),\quad p_n=(1/a)\log E.$$

For any other starting point the author shows that p_n and h_n oscillate infinitely as $n\to\infty$, taking indefinitely small as well as indefinitely large values, reach maxima only above and minima only below their equilibrium values, and are never simultaneously near their equilibrium values. Also

$$\overline{\lim}\,(ap_n)^{1/n}=\overline{\lim}\,(ah_n)^{1/n}=E^{\frac{1}{2}},$$

$$\overline{\lim}\,(\log 1/ap_n)^{1/n}=\overline{\lim}\,(\log 1/ah_n)^{1/n}=E^{\frac{1}{2}},$$

(when the logarithms are positive) and

$$\underline{\lim}\,(1/n)(p_1+p_2+\cdots+p_n)=\log E/2a,$$

$$\underline{\lim}\,(1/n)(h_1+h_2+\cdots+h_n)=\log E/a(E-1).$$

W. E. Milne (Corvallis, Ore.).

Hamilton, Hugh J. On monotone and convex solutions of certain difference equations. Amer. J. Math. 63, 427–434 (1941). [MF 4162]

The author shows how conditions insuring uniqueness or existence of a continuous convex solution $U(x)$ of a difference equation

$$\sum_{\mu=0}^{n} a_\mu U(x+\mu)=G(x)$$

yield conditions for the existence or uniqueness of monotone solutions $u(x)$ of the equation

$$\sum_{\mu=0}^{n} a_\mu u(x+\mu)=G^+(x),$$

where $G^+(x)$ is the right-hand derivative of G, and vice versa. Using sufficient conditions for uniqueness of monotone or continuous convex solutions given by F. John [Acta Math. 71, 175–189 (1939); cf. these Rev. 1, 72] the author obtains a number of more general conditions. *F. John.*

Thielman, H. P. On the convex solution of a certain functional equation. Bull. Amer. Math. Soc. 47, 118–120 (1941). [MF 3823]

The author proves that the only solution of the functional equation

$$1/f(x+a)=x^pf(x),\qquad x>0;\, a>0;\, p>0,$$

which is convex (or monotone non-increasing, for sufficiently large x) is given by

$$f(x)=\left[\frac{\Gamma(x/2a)}{\sqrt{2a}\,\Gamma((x+a)/2a)}\right]^p.$$

The special case $p=1$ had been proved by A. E. Mayer [Acta Math. 70, 57–62 (1938)]. *F. John.*

John, Fritz. Discontinuous convex solutions of difference equations. Bull. Amer. Math. Soc. 47, 275–281 (1941). [MF 4175]

A. E. Mayer [Acta Math. 70, 57–62 (1938)] and H. P. Thielman [cf. the preceding review] have shown that the equation $f(x+1)\cdot f(x)=x^{-p}$ ($x>0$, $p>0$) has only one convex solution (convex in the general sense of Jensen). This is now greatly generalized with the result that a difference equation $\prod_{k=0}^{n}(f(x+k))^{a_k}=g(x)$ ($x>0$) has at most one convex solution provided the following conditions are verified: a_k are real, $\sum_0^n a_kx^k=0$ has only simple roots of absolute value 1, $a_n>0$, $\sum_0^n a_k\neq 0$, $g(x)\neq 0$ and continuous for $x>0$,

$$\lim_{x\to\infty}(\log|g(x)|)/x=0,\quad \lim_{x\to\infty}(\log|g(x)|)/\log x\neq\sum_0^n a_k.$$

The proof is based on the author's earlier work [Acta Math. 71, 175–189 (1939); cf. these Rev. 1, 72] and makes very effective use of F. Bernstein's analysis of convex functions and of a simple algebraic result of E. Meissner [Math. Ann. 64 (1907) and 70 (1911), respectively]. Further results [the author's theorems are stronger] are to the effect that the equation $\sum_{k=0}^{n}a_kf(x+k)=0$ has or does not have discontinuous convex solutions according as $\sum_0^n a_kx^k=0$ has or does not have a positive root. In the first case discontinuous solutions are constructed by using a Hamel's base for real numbers. *I. J. Schoenberg* (Philadelphia, Pa.).

Sbrana, Francesco. Sopra alcune ricerche riguardanti il calcolo degli operatori funzionali. Pont. Acad. Sci. Acta 3, 73–78 (1939). [MF 4097]

In order to obtain the result of the operation

$$u(x,t)=e^{-bt}(\operatorname{ch} xq-\operatorname{sh} xq\cdot\coth aq)e^{bt}f(t),\qquad 0<x<a,$$

when q denotes the operator $(D_t^2-b^2)^{\frac{1}{2}}$ the author makes use of Giorgi's result that if $F(t)$ is zero for $t<0$ then $e^{-xq}F(t)=0$ for $t<x$ but for $t>x$

$$e^{-xq}F(t)=F(t-x)+bx\int_{-\infty}F(s)I_1(T)ds/T,$$

where $T^2=(t-s)^2-x^2$. It is found in agreement with Serini that for $t<2a$, $u(x,t)=e^{-bt}e^{-xq}e^{bt}f(t)$. With the aid of Heaviside's unit function it is shown that a solution of the partial differential equation $au_{xx}+bu_{yy}+cu_{xy}+du_x+eu_y=f(x,y)$ may be expressed in the form

$$u(x,y)=G(x-x_0,y-y_0)f(x_0,y_0)dxdy,$$

where

$$4\pi^2G(x,y)=-\int dm\int .e^{mx+ny}dn/(am^2+bn^2+cmn+dm+en).$$

Solutions of different types may be obtained by using different forms of path relative to the singularities of the integrand. *H. Bateman* (Pasadena, Calif.).

Plessner, A. Über die Einordnung des Heaviside'schen Operationskalküls in die Spektraltheorie maximaler Operatoren. C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 10–12 (1940). [MF 3539]

Heaviside's calculus of operations is obtained by considering conditions in appropriate Hilbert spaces. The functions f in these spaces obtain from the unit function 1 by an operation $f=F(id/dt)1$. Here the operator function F of the maximal operator id/dt is being used. This transformation from f to F yields the usual formulas, and permits a vigorous justification of the procedures which lead to them. *J. von Neumann* (Princeton, N. J.).

Integral Equations

Kneschke, A. Wechselbeziehungen zwischen Differentialgleichungen und Integralgleichungen. Deutsche Math. 5, 384–393 (1941). [MF 4249]

The author proves, by well-known methods, that a par-

ticular type of Volterra integral equation of the second kind is equivalent to a Cauchy integration problem for an nth order ordinary linear differential equation. He also establishes the corresponding result that a particular type of Fredholm integral equation of the second kind is equivalent to a Lagrange integration problem for such a differential equation. *W. T. Reid* (Chicago, Ill.).

Miranda, Carlo. Su alcuni sviluppi in serie procedenti per funzioni non necessariamente ortogonali. Pont. Acad. Sci. Acta **3**, 1–4 (1939). [MF 4093]

In this note the author gives a formal discussion of expansion theorems in terms of the characteristic solutions of an integral equation whose kernel, although symmetric, involves the characteristic parameter. *W. T. Reid.*

Dressel, F. G. A Stieltjes integral equation. Bull. Amer. Math. Soc. **47**, 79–83 (1941). [MF 3815]

The integral equation considered is

$$F(x)=M(x)+\lambda\int_0^1 H(x, y)dF(y),$$

where the integral is of the Young-Stieltjes type. By assuming that $M(x)$ and $H(x, y)$ are absolutely continuous with respect to the bounded increasing function $g(x)$ which is continuous on the right, except perhaps at $x=0$, the solution is reduced to that of

$$v(x)=m(x)+\lambda\int_0^1 h(x, y)v(y)dg(y),$$

where $m(x)$ and $h(x, y)$ are derived numbers of $H(x)$ and $M(x, y)$, respectively, with respect to $g(x)$. If $g(x)$ is such that, for $x_1<x_2$, $g(x_1)<g(x_2)$, then the transformation $y=\beta(s)=\overline{\lim}\, E_y(s\geqq g(y))$ makes the results for the Fredholm equation on functions of L^2 available. The author considers also the integral equation

$$f(x)=m(x)+\lambda\int_0^1 f(y)d_yK(x, y)$$

of Fischer [Ann. of Math. (2) **25**, 124–158 (1923)] and shows that Fischer's condition

$$V_yK=\text{l.u.b.}\Big[\sum_{i=1}^{n}|K(x_i, y_i)-K(x_i, y_{i-1})|\Big]<\infty$$

($x_1, \cdots, x_n$ on $(0, 1)$; $0=y_1<y_2 \cdots <y_n=1$) implies the existence of a function $A(y)$ monotonic on $(0, 1)$, such that

$$|K(x, y_1)-K(x, y_2)|\leqq|A(y_1)-A(y_2)|$$

for all y_1 and y_2 on $(0, 1)$, and consequently the Stieltjes integral equation is reducible to a Fredholm equation.
T. H. Hildebrandt (Ann Arbor, Mich.).

Krein, M. G. On "loaded" integral equations the distribution functions of which are not monotonic. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 88–103. (Russian) [MF 3507]

The author considers kernels $K(x, s)$ such that (A) $|K(x, s)|<M$ $(a\leqq x, s\leqq b)$ and (B) $K(x, s)$ is continuous in x (in s), on (a, b), for every s on (a, b) (x on (a, b)); the equations considered are

$$(1)\qquad \varphi(x)-\lambda\int_a^b K(x, s)\varphi(s)d\sigma(s)=f(x),$$

where $\sigma(x)$ is of bounded variation. It is known that a Fredholm-type theory will hold for (1), under (A), (B). A theory of Schmidt-type is on hand when $K(x, s)$ is symmetric, (A) and (B) hold and $\sigma(x)$ is "essentially" monotone; the author demonstrates that the latter condition may be replaced by the requirement that

$$\int_a^b\int_a^b K(s, x)d\tau(x)d\tau(s)\geqq 0$$

for all functions $\tau(x)$ of bounded variation. An essential role in the demonstrations is played by Grommer's theorem, giving conditions under which a meromorphic function has real and simple poles only, and by an extension of a theorem of F. Riesz, relating to approximation in a closed set $E\subset(a, b)$ of functions, continuous for $a\leqq x\leqq b$, by sums of the form $c_1\varphi_1(x)+\cdots+c_n\varphi_n(x)$, where the $\varphi_i(x)$ are prescribed functions continuous on (a, b). Equation (1) is a special instance of certain equations studied by J. Radon [1919] and N. Gunther [Trav. Inst. Phys.-Math. Stekloff **1**, 1–494 (1932)]. *W. J. Trjitzinsky* (Urbana, Ill.).

Vecoua, I. Sur les équations intégrales linéaires singulières contenant des intégrales au sens de la valeur principale de Cauchy. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 327–330 (1940). [MF 3557]

Consider the integral equation

$$(1)\qquad \alpha(\sigma)\phi(\sigma)-\lambda\int_S \frac{K(\sigma, \tau)}{\tau-\sigma}\phi(\tau)d\tau=f(\sigma),$$

where S is a simple closed curve (in the complex plane) of bounded curvature at each point and contains the origin in its interior. σ, τ are complex coordinates on S; $K(\sigma, \tau)$, $\alpha(\sigma)$ and $f(\sigma)$ satisfy a Hölder condition and the integral is understood in the sense of a Cauchy principal value. The problem of equivalence of (1) to a Fredholm integral equation with solutions satisfying a Hölder condition has been treated in its essentials by Michlin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 315–317 (1939); these Rev. **2**, 100]. The basic ideas occur in Carleman [Ark. Mat. Astr. Fys. **16** (1922)]. The author modifies Michlin's approach in part and shows the equivalence problem may be reduced to the Riemann problem of finding two functions, holomorphic inside and outside S, respectively, whose boundary values on S satisfy a Hölder condition and are connected by a simple relation. The known explicit solutions of this Riemann problem imply Michlin's equivalence theorems.
D. G. Bourgin (Urbana, Ill.).

Collatz, L. Schrittweise Näherungen bei Integralgleichungen und Eigenwertschranken. Math. Z. **46**, 692–708 (1940). [MF 3376]

Inequalities between the eigenvalues λ of the integral equation $\varphi(s)=\lambda\int K(s, t)\varphi(t)dt$ are obtained in terms of the quantities $a_n=\int F_m(s)F_{n-m}(s)ds$, where F_m is defined by iterations: $F_m(s)=\int K(s, t)F_{m-1}(t)dt$, F_0 arbitrary. A generalization of Temple's inclusion theorem is developed. In detailed numerical examples the efficiency of the obtained estimates is tested. *K. Friedrichs* (New York, N. Y.).

Schröder, Kurt. Über die Prandtlsche Integro-Differentialgleichung der Tragflügeltheorie. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 16, 35 pp. (1939). [MF 3779]

The equation in question is

$$(*)\qquad G(y)=L(y)\left\{a(y)-\frac{1}{4\pi}P\int_{-1}^{+1}\frac{dG(\sigma)}{d\sigma}\frac{d\sigma}{y-\sigma}\right\},$$

where P denotes Cauchy's principal value of the integral. Using results of a previous paper [S.-B. Preuss. Akad. Wiss. **1938**, 345–362] the author reduces (*) to an integral equation of the second kind. Emphasis is laid on the precise conditions which make the transformation possible. Some properties of the solution are deduced directly from the integral equation; using a method of Enskog [Math. Z. **24**, 670–683 (1926)] the author obtains series expansions of the solution and an appraisal of the error committed when only a finite number of terms are taken into consideration.

W. Feller (Providence, R. I.).

Jaeckel, Karl. Auflösung der Prandtlschen Zirkulationsgleichung durch trigonometrische Reihen. Luftfahrtforschung **17**, 47–53 (1940). [MF 3957]

The author begins with the Prandtl integro-differential equation for the circulation distribution of a monoplane lifting line and obtains a solution by expressing the circulation and other functions as Fourier series. The resulting system of equations and the scheme proposed for its numerical solution are equivalent to the results of I. Lotz [Z. Flugtech. Motorluftschiffahrt **1931**]. By a rather lengthy analysis, which avoids the improper integral of Prandtl's equation, and which makes use of a theorem of Hilbert, it is shown that the assumption of quadratic integrability of the given functions (chord, twist) is sufficient to insure unique solubility of the system for the Fourier coefficients. In a supplement [Luftfahrtforschung **17**, 81 (1940)] the author points out the similarity of his methods to those of Lotz and states that his results assure the validity of the Lotz procedure in all practical cases. *W. R. Sears* (Inglewood, Calif.).

Birkhoff, George D. On drawings composed of uniform straight lines. J. Math. Pures Appl. (9) **19**, 221–236 (1940). [MF 4622]

The straight lines are assumed to be large in number, indefinite in extent, constant in width and intensity with the surface density of lead, additive. Let $F(r,\theta)$ be the density function (in polar coordinates). This is dependent on the distribution function $f(s,\varphi)$ along the straight line, where s, φ are the normal parameters of the straight line. Then $s=r\sin(\varphi-\theta)$ and $F(r,\theta)=\int_0^{2\pi}f(r\sin(\varphi-\theta),\varphi)d\varphi$. The paper centers in determining $f\geqq 0$ for a given F. In the case of circular symmetry $F(r,\theta)$ and $f(s,\varphi)$ are independent of θ and φ and the equation $F(r)=\int_0^{2\pi}f(r\sin u)du$ is transformed into the Abel equation $F(r)=4\int_0^r f(s)ds/(r^2-s^2)^{\frac{1}{2}}$, leading to a determination of $f(s)$. In the general case F and f are assumed to be asymptotically representable by a power series in the corresponding rectangular coordinates. This suggests utilizing the formal Fourier expansions

$$F(r,\theta)\sim\tfrac{1}{2}F_0(r)+\sum_{m=1}^{\infty}[F_m(r)\cos m\theta+G_m(r)\sin m\theta],$$

$$f(s,\varphi)\sim\tfrac{1}{2}f(s)+\sum_{m=1}^{\infty}[f_m(s)\cos m\varphi+g_m(s)\sin m\varphi],$$

the substitution of which leads to a series of integral equations $H_m(s)=\int_0^{2\pi}h_m(r\sin u)e^{-imu}du$, where $H_m=F_m+iG_m$ and $h_m=f_m+ig_m$, which can be transformed into equations similar to the Abel equation. It is then concluded that under the hypotheses on F and f a unique distribution function $f\geqq 0$ exists corresponding to a given density function F. The case where expansion in power series is replaced by continuity is handled by approximation. The case where f is not restricted to being positive but is bounded has a counterpart in the use of uniform erasure.

T. H. Hildebrandt (Ann Arbor, Mich.).

Functional Analysis, Ergodic Theory

Grunblum, M. M. et Gourevitch, L. A. Sur une propriété de la base dans l'espace de Hilbert. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 289–291 (1941). [MF 4381]

A biorthogonal sequence is a basis if it is complete and if the partial sums correspond to linear transformations which are uniformly bounded. These partial sums are essentially finite dimensional transformations, whose bounds can be readily obtained. *F. J. Murray* (New York, N. Y.).

Gantmacher, Vera und Šmulian, Vitold. Über schwache Kompaktheit im Banachschen Raum. Rec. Math. [Mat. Sbornik] N.S. **8** (50), 489–492 (1940). (Russian. German summary) [MF 3745]

The following result is proved: let E be a Banach space and $K\subset E$ a set weakly compact in E. Then the unit sphere of the conjugate space is weakly compact relative to K; in other words, each sequence of linear functionals $\{f_n\}$, $|f_n|\leqq 1$, contains a subsequence $\{f_{n_i}\}$ which converges to a linear functional f_0, $|f_0|\leqq 1$, in the sense $f_{n_i}(x)\rightarrow f_0(x)$ for all $x\varepsilon K$. *J. D. Tamarkin* (Providence, R. I.).

Vulich, B. Z. Linear spaces with given convergence. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 40–63 (1940). (Russian) [MF 3302]

Let $\mathfrak{L}$ be a linear space in which there is defined "γ-convergence to 0," $x_n\xrightarrow{\gamma}0$, satisfying the postulates: (1) $x_n\xrightarrow{\gamma}0$, $y_n\xrightarrow{\gamma}0$ implies $x_n+y_n\xrightarrow{\gamma}0$; (2) $x_n\xrightarrow{\gamma}0$ implies $\alpha x_n\xrightarrow{\gamma}0$ for every real α; (3) $\alpha_n\rightarrow 0$ (α_n real) implies $\alpha_n x\xrightarrow{\gamma}0$ for every x; (4) if $x_n=x$ and $x_n\xrightarrow{\gamma}0$, then $x=0$; (5) if $x_n\xrightarrow{\gamma}0$ then every subsequence also γ-converges to 0; (6) if $x_n\xrightarrow{\gamma}0$, $y_n\xrightarrow{\gamma}0$ and $z_n=x_n$ for $n=2m-1$, $z_n=y_n$ for $n=2m$, then $z_n\xrightarrow{\gamma}0$; (7) if $x_n\xrightarrow{\gamma}0$ and $\{y_n\}$ is obtained from $\{x_n\}$ by any rearrangement of the sequence $\{x_n'\}$, where $x_1'=\cdots=x'_{k_1}=x_1$, $x'_{k_1+1}=\cdots=x'_{k_2}=x_2,\cdots$; $k_1<k_2<\cdots$, then $y_n\xrightarrow{\gamma}0$. It is said that $x_n\xrightarrow{*}0$ if every subsequence of $\{x_n\}$ contains another subsequence which γ-converges to 0. It is said that $x_n\xrightarrow{\gamma}x$ or $x_n\xrightarrow{*}x$ if, respectively, $x_n-x\xrightarrow{\gamma}0$ or $x_n-x\xrightarrow{*}0$. Introducing topology in $\mathfrak{L}$ in the usual way the author shows that the convergence in sense of this topology is equivalent to $*$-convergence. The space $\mathfrak{L}$ so topologized is called space $\mathfrak{L}\mathfrak{T}$. Among other results it is proved that a necessary and sufficient condition that a space $\mathfrak{L}\mathfrak{T}$, in which γ-convergence and $*$-convergence are identical, be metrizable to a space F, so that γ-convergence would coincide with the convergence in metric, is that the space should satisfy the first denumerability axiom of Hausdorff [an analogous theorem for topological groups was proved by G. Birkhoff [Compositio Math. **3**, 427–430 (1936)]]. In order that a space $\mathfrak{L}\mathfrak{T}$ could be metrized to a Banach space it is necessary and sufficient that there should exist a bounded convex neighborhood of the zero element [for linear topological spaces this was proved by Kolmogoroff [Studia Math. **5**, 29–33 (1934)]]. In case the γ-convergence does not coincide with the $*$-convergence the author introduces the "space of complexes" $(x_1,x_2,\cdots,x_k)$ and generalizes in a suitable way the notion of the norm to a "norm of a complex" and k-convergence to a "k-convergence." He solves an analogous problem of metrization in case the γ-convergence coincides with this k-convergence. Finally, some applications to semiordered spaces are indicated. *J. D. Tamarkin.*

Day, Mahlon M. Some more uniformly convex spaces. Bull. Amer. Math. Soc. **47**, 504–507 (1941). [MF 4546]

Earlier results of Boas [Bull. Amer. Math. Soc. **46**, 304–311 (1940); these Rev. **1**, 242] and Day [Bull. Amer. Math. Soc. **47**, 313–317 (1941); these Rev. **2**, 221] are completed here by the following theorem: let B_i ($i=1, 2, 3, \cdots$) be a sequence of uniformly convex Banach spaces, and let B be the Banach space of sequences $b=\{b_i\}$ ($b_i \epsilon B_i$). The norm in B is defined by $\|b\|=[\sum_{i=1}^{\infty}\|b_i\|^p]^{1/p}$, where p is a fixed real greater than one. Then B is uniformly convex if and only if the B_i have a common modulus of convexity.

J. A. Clarkson (Philadelphia, Pa.).

Yosida, Kôsaku and Fukamiya, Masanori. On regularly convex sets. Proc. Imp. Acad. Tokyo **17**, 49–52 (1941). [MF 4450]

Let E be a Banach space and $\bar{E}$ its conjugate space. A set $F \subseteqq \bar{E}$ is said to be regularly convex (Krein-Smulian) if for every $g \epsilon \bar{E}-F$ there is an $x_0 \epsilon E$ with $\sup_{f \epsilon F} f(x_0) < g(x_0)$. For $X \subseteqq E$ let X^* consist of all $f \epsilon \bar{E}$ for which $\sup_{x \epsilon X} f(x) \leqq 1$, and for $F \subseteqq \bar{E}$ let F' consist of all $x \epsilon E$ such that $\sup_{f \epsilon F} f(x) \leqq 1$. Then the sets X^* are precisely those regularly convex sets $F \subseteqq \bar{E}$ which contain 0. Dually, the sets F' are precisely those strongly closed convex sets in E which contain 0. In order that a regularly closed convex set F contain 0 as an inner point it is necessary and sufficient that the closed convex set F' be bounded. The dual theorem is also given. The paper contains a new proof of the Krein-Milman result on existence of extreme points for bounded regularly convex sets F.

N. Dunford (New Haven, Conn.).

Šmulian, V. Sur quelques propriétés géométriques de la sphère dans les espaces linéaires semi-ordonnés de Banach. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 394–398 (1941). [MF 4378]

Let K be a subset of a linear normed space E having the properties: (1) If $x \epsilon K$, $\lambda x \epsilon K$ for $\lambda \geqq 0$. (2) If $x, y \epsilon K$, $x+y \epsilon K$. (3) For each $x \epsilon E$, $x=x_1-x_2$, where $x_1, x_2 \epsilon K$. (4) K is closed. Let Q be the elements of K of norm not greater than 1. Necessary and sufficient conditions for the regularity (reflexiveness) of E are given by: (1) For every transfinite sequence $\{x_\xi\} \subset Q$ there exists $x_0 \epsilon Q$ such that $\underline{\lim}\, f(x_\xi) \leqq f(x_0) \leqq \overline{\lim}\, f(x_\xi)$ for all $f \epsilon \bar{E}$. (2) A transfinite decreasing sequence of closed and convex sets in Q has a non-empty intersection. (3) Every partially ordered set $\{x_\alpha\} \subset Q$ with $\lim \|\frac{1}{2}(x_{\alpha'}+x_{\alpha''})\|=1$ which converges weakly to an element x_0. In case the unit sphere of E is weakly compact other conditions for regularity are obtained involving the elements of K of unit norm. Finally, conditions (3) and (4) on K are changed and a sufficient condition for regularity is obtained in terms of uniform strong differentiability of the norm on a certain subset of K. *J. V. Wehausen* (Columbia, Mo.).

Udin, A. I. Some geometric questions in the theory of linear semi-ordered spaces. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 64–83 (1940). (Russian) [MF 3303]

The author considers a semi-ordered space K_4 which satisfies the postulates: (1) $y>0$ excludes $y=0$; (2) $y_1>0$, $y_2>0$ implies $y_1+y_2>0$; (3) for each y there exists $\bar{y}=\sup(y, 0)$; (4) if $y>0$ and $\lambda>0$ is real, then $\lambda y>0$. A finite-dimensional semi-ordered space is said to be singular if there exists a negative element which is a limit of a sequence of positive elements; a general linear semi-ordered space is non-singular if every linear finite-dimensional subspace is non-singular. Two main results are established: (I) Every n-dimensional semi-ordered non-singular space K_4 is isomorphic to the n-dimensional Euclidean space R_n. (II) Let F_α, $\alpha=1, 2, \cdots, m$, be expressions containing real-valued variables $x_1, \cdots, x_k$, combined by a finite number of additions, subtractions, multiplications by a real constant, and taking sup or inf. Then, if K_4 is non-singular, and if $F_\alpha=0$, $\alpha=1, 2, \cdots, (m-1)$, implies $F_m=0$, then the same will be true if $x_1, \cdots, x_k$ are replaced by elements of K_4. The proofs are based on certain geometric interpretations and properties of spaces K_4. *J. D. Tamarkin* (Providence, R. I.).

Vernikoff, I., Krein, S. et Tovbin, A. Sur les anneaux semi-ordonnés. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 785–787 (1941). [MF 4477]

Let R be a partially ordered ring with unit element and real multipliers. The problem considered in this note is that of finding conditions on R which will insure that it is (ring and order) isomorphic to the set of all real continuous functions over some bicompact Hausdorff space. The proof is briefly outlined that the following three conditions are sufficient (θ, e stand for the zero and unit elements of R): (A) For each $x \epsilon R$ there is a real positive t such that $-te<x<te$. (B) Such t's have a positive lower bound for each $x \epsilon R$. [It seems to the reviewer that the trivial exception $x=\theta$ must be made here.] (C) If ρ_x is the inf of those real t's for which $x<te$, then $x \leqq \rho_x e$. The proof employs the ideals in R, and their quotients. No assumption is made that R is commutative. *J. A. Clarkson* (Philadelphia, Pa.).

Šilov, G. On the extension of maximal ideals. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 83–84 (1940). [MF 3696]

For R a normed ring, the author proves the following two results: Th. 1. If there exists a constant $K>0$ such that $\|xy\| \geqq K\|x\| \cdot \|y\|$ for arbitrary $x, y \epsilon R$, then R is the field of complex numbers. Th. 2. Let R contain a subring R_1, which contains with every x its complex conjugate. Then every maximal ideal $M_1 \subset R_1$ is contained in a certain maximal ideal $M \subset R$. *J. A. Clarkson* (Philadelphia, Pa.).

Paxson, E. W. Sur un espace fonctionnel abstrait. Revista Ci., Lima **42**, 817–821 (1940). [MF 4345]

The function space F whose elements are continuous functions defined on the unit interval, with values in a linear topological space L, is topologized by taking as a neighborhood of the zero element of F the set of all elements of F whose functional values lie in any convex set of L which contains a neighborhood of the zero element of L. It is proved that F is a linear topological space in the sense of von Neumann if L is, and it is stated that the set of all elements of F whose functional values lie in a fixed bicompact set of L is a bicompact set of F. *O. Frink*

Paxson, E. W. Les équations différentielles dans les espaces linéaires et topologiques. Revista Ci., Lima **42**, 823–826 (1940). [MF 4346]

Let F be the space of continuous functions defined over the unit interval and taking values in a linear topological space L. Suppose that $f(x, y)$ is a continuous function whose values lie in a convex and bicompact set A of L, containing the zero element θ of L, where the real variable x ranges over the unit interval, and the abstract variable y ranges over a set L_0 of L containing θ. The author sketches a proof of the theorem that there exists at least one function $y(x)$ of F such that $y'(x)=f(x, y(x))$ and $y(0)=\theta$, where $y'(x)$ is the abstract valued derivative defined by A. D. Michal and

the author. The proof is based on the Tychonoff fixed point theorem, and on a result of the author concerning a method of topologizing the space F [see the preceding review]. To complete the proof the additional hypothesis that $A \subset L_4$ seems to be required. No mention is made of the fact that the Tychonoff theorem applies only to continuous transformations. The author points out that this type of proof is non-effective in character, and implies nothing concerning the uniqueness of the solution. *O. Frink.*

Hyers, D. H. On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U. S. A. **27**, 222–224 (1941).

A transformation $f(x)$ from a Banach space E to another E' is δ-linear if $\|f(x+y)-f(x)-f(y)\|<\delta$ for all $x, y \epsilon E$. The following theorem is proved. Let $f(x)$ be a δ-linear transformation of E into E'. Then $l(x)=\lim f(2^n x)/2^n$ exists for each x in E, $l(x)$ is linear, and the inequality $\|f(x)-l(x)\|\leqq\delta$ is true for all x in E. Moreover, $l(x)$ is the only linear transformation satisfying this inequality. If, in addition, it is assumed that $f(x)$ is continuous at at least one point, then $l(x)$ is continuous everywhere in E.

J. V. Wehausen (Columbia, Mo.).

Wong, Y. K. On biorthogonal matrices. Bull. Amer. Math. Soc. **47**, 424–431 (1941). [MF 4532]

The biorthogonality concerns two matrices $\kappa(p_1p_2)=\kappa^{12}$ and $\varphi(p_2p_1)=\varphi^{21}$ on the products $\mathfrak{P}_1\mathfrak{P}_2$ and $\mathfrak{P}_2\mathfrak{P}_1$ of two general ranges to a number system A, having a conjugate but in which multiplication is not necessarily commutative. The integration processes used in the definition of J^1 and J^2 are derived from the positive Hermitian matrices $\epsilon^1=\epsilon_1(p_1p_1)$ and $\epsilon^2=\epsilon_2(p_2p_2)$ [see Moore, General Analysis, Part II, pp. 84 ff.] and corresponding classes $\mathfrak{M}(\epsilon^1)$ and $\mathfrak{M}(\epsilon^2)$ give rise to the condition of biorthogonality $J^1\mu_1\nu_1=J^2(J^1\bar{\mu}_1\kappa^{12}, J^1\varphi^{21}\mu_1)$, where μ_1 and ν_1 are of $\mathfrak{M}(\epsilon^1)$ and such that $J^1\bar{\mu}_1\kappa^{12}$ and $J^1\varphi^{21}\nu_1$ are of $\mathfrak{M}(\epsilon^2)$, that is, so that J^2 is effective, a generalization of the Parseval equality. More generality is obtained by assuming that κ^{12} is by columns of $\mathfrak{M}(\epsilon_0^1)$ and φ^{21} by rows conjugate of $\mathfrak{M}(\epsilon_1^1)$, where ϵ_0^1 and ϵ_1^1 are idempotent, or unitary relative to ϵ^1, that is, $J^1\epsilon_i^1\epsilon_i^1=\epsilon_i^1$ ($i=0, 1$). Connections between various biorthogonality conditions and their relation to the classes defined by ϵ^1 and ϵ^2 are given. Among other results conditions under which the expansion $\mu_1=J^2\kappa^{12}J^1\varphi^{21}\mu_1$ is valid, and the reciprocal relation $\epsilon^2=J^1\varphi^{21}\kappa^{12}$ holds, are derived. [Cf. also the author's paper, Bull. Amer. Math. Soc. **46**, 352–355 (1940); these Rev. **2**, 103.]

T. H. Hildebrandt (Ann Arbor, Mich.).

Alexandroff, A. D. Additive set-functions in abstract spaces. Rec. Math. [Mat. Sbornik] N.S. **8** (50), 307–348 (1940). (English. Russian summary) [MF 3467]

This paper consists of the introduction and the first chapter of a monograph of six chapters to appear in this same journal under the same title. The central problem is that of establishing connections between linear functionals in the space of continuous functions of a general space R and additive set functions in R. On one hand the notion of space is generalized in that it is only demanded of the closed sets that the countable intersection of closed sets be closed; this is motivated by measure-considerations. (It is obvious, of course, that by using the "closed" sets of a "space" R as an (arbitrarily) multiplicative base one gets an ordinary topological space; this is called the topological prolongation of R.) On the other hand the additive set functions $\mu(E)$ on R are specialized as follows: each μ is bounded, defined on the field generated by the closed sets of R, and regular (that is, given $\epsilon>0$ there exists an open set G containing E such that $|\mu(E)-\mu(G)|<\epsilon$); a set function of this sort is called a charge. A charge is more general than a measure for it need be neither positive nor totally additive.

In Chapter I the definition of a "space" is given and special cases are discussed. For some reason the usual designation perfectly normal (Čech, Vedenisoff) to mean a space in which each closed set is the set of zeros of some continuous function is replaced by completely normal, although the latter term is generally used for something slightly weaker. Attention is mainly devoted to bicompact spaces and bicompact extensions of spaces, many of the theorems being generalizations of known results. There is a close connection with work of Čech and Wallman [Ann. of Math. (2) **38**, 823–844 (1937) and **39**, 112–126 (1938), respectively]. The introduction contains rather complete abstracts of each of the six chapters. *H. Wallman* (Chapel Hill, N. C.).

Bochner, S. and Phillips, R. S. Additive set functions and vector lattices. Ann. of Math. (2) **42**, 316–324 (1941). [MF 3687]

This paper contains an elementary proof of the theorem of Bochner asserting that in the Banach space of additive set functions, which are absolutely continuous with respect to a given Jordan content, the step functions are dense. The basic notions of the paper are those of the vector lattice and of the projection on a normal subspace. Some analogies between vector lattices and set functions are given.

N. Dunford (New Haven, Conn.).

Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 465–478 (1940). (Russian. French summary) [MF 4085]

Let $\{E\}$ be a system of subsets of an abstract set X such that $\{E\}$ contains the complement of any of its sets and the sum of a finite or denumerable number of them. A function $\varphi(E)$ defined on $\{E\}$ with values in Euclidean R^n is said to be a completely additive vector-function if, for any system of disjoint sets $E_1, E_2, \cdots \epsilon\{E\}$, the lengths of the vectors $\varphi(E_n)$ form a convergent series and $\varphi(\sum E_n)=\sum\varphi(E_n)$. $\varphi(E)$ is said to have a saltus on the set $E\epsilon\{E\}$ if, for any set $E'\subset E$ and belonging to $\{E\}$, either $\varphi(E')=0$ or $\varphi(E')=\varphi(E)\neq0$. φ is identically zero on a set $E\epsilon\{E\}$, $\varphi(E)\equiv0$, if, for any $E'\epsilon\{E\}$, $\varphi(E'\cdot E)=0$. The collection of sets $E'\epsilon\{E\}$ with $\varphi(E-E')\equiv\varphi(E'-E)\equiv0$ is called the metric type of the set E, denoted by E^*. One may define $\varphi(E^*)=\varphi(E)$. The following theorems are proved. (1) The set of values K of any completely additive vector-function φ is closed. (2) If φ has no saltuses, K is convex. (3) If φ has no saltuses and if p is any element of K, then $\varphi(E^*)=p$ has continuum-many different solutions if p is interior to some line segment of K, otherwise a unique solution.

J. V. Wehausen (Columbia, Mo.).

Krein, M. Propriétés fondamentales des ensembles coniques normaux dans l'espace de Banach. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 13–17 (1940). [MF 3591]

Let E be a linear normed space, $\bar{E}$ its conjugate and $K\subset E$ a normal conical set characterized by the following conditions: (1) If $x\epsilon K$, then $\lambda x\epsilon K$, $\lambda\geqq0$. (2) If $x\epsilon K$, $y\epsilon K$, then $x+y\epsilon K$. (3) If $x\epsilon K$, $x\neq0$, then $-x\bar{\epsilon}K$. (4) There exists a $\delta>0$ such that for any two e_1, e_2 of K, $|e_i|=1$, $|e_1+e_2|>\delta$.

Let Q be the set of all $x\epsilon E$ such that there exist $u\epsilon E$, $v\epsilon E$, $|u|\leqq 1$, $|v|\leqq 1$ and $u<x<v$. [We say $x<y$ (or $y>x$) if $x\neq y$ and $y-x\epsilon K$. We also say $g<f$ (g, $f\epsilon \bar{E}$) if $g\neq f$ and $g(x)\leqq f(x)$ for all $x\epsilon K$.] The author proves that boundedness of Q is necessary and sufficient in order that a set satisfying (1), (2), (3) be a normal conical set. For any $x\epsilon E$ let $|x|_Q=\inf t$, $t>0$, $x\epsilon tQ$. Let E_Q be the linear normed space obtained from E by identifying elements x and y for which $(x-y)_Q=0$. Every linear functional $f\epsilon \bar{E}_Q$ determines a linear functional in $\bar{E}$. A necessary and sufficient condition that a linear functional $f\epsilon\bar{E}$ admit of a decomposition $f=g-h$, $g\geqq 0$, $h\geqq 0$, is that f be determined by a functional on E_Q, in other words, be bounded on E_Q. In order that every $f\epsilon\bar{E}$ could be so decomposed it is necessary and sufficient that the set K satisfying (1), (2), (3) would be a normal conical set. Among several other results obtained by the author we mention only the two following. Let K be a normal conical set. If the series $\sum u_n$, $u_1+u_2+\cdots+u_i>0$, $i=1, 2, \cdots$, converges weakly to an element v, then it converges strongly to v, no matter what is the order of terms. Let K be a normal conical set, $u_i>0$, $i=1, 2, \cdots$. If the power series $u_0+u_1\lambda+u_2\lambda^2+\cdots$ has a finite radius of convergence ρ, then the point $\lambda=\rho$ is singular for this power series. *J. D. Tamarkin* (Providence, R. I.).

Krein, M. Sur la décomposition minimale d'une fonctionnelle linéaire en composantes positives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 18–22 (1940). [MF 3592]

In the previous paper [see the preceding review] the author introduced the notion of a normal conical set K and considered decompositions $f=g-h$ of linear functionals, where $g\geqq 0$, $h\geqq 0$. In the present note the author is interested in existence of minimal decompositions, that is, such that whenever $f=g'-h'$, $g'\geqq 0$, $h'\geqq 0$, then $g'>g$, $h'>h$. Now it is assumed that the normal conical set K has interior points and it is proved that a necessary and sufficient condition in order that every $f\epsilon\bar{E}$ admit of a minimal decomposition is that the inequalities $x\ll u$, $x\ll v$, $u\gg 0$, $v\gg 0$ imply the existence of a $y\gg 0$ such that $x\ll y\ll u$, $x\ll y\ll v$. Here $x\ll y$ means that $y-x\epsilon K$ and is an interior element of K. *J. D. Tamarkin* (Providence, R. I.).

Krein, Mark. On a special ring of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 355–359 (1940). [MF 3964]

Let Q be a compact metric space and let R be the ring determined by the positive definite Hermitian kernels $\phi(s, t)$, s, $t\epsilon Q$. The method of Gelfand [C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 430–432 (1939); **25**, 570–574 (1939); these Rev. **1**, 330] is applied to this ring R. It leads to the results: (A) If $\phi\epsilon R$ and $f(z)$ is a function analytic in the region of the values $z=\phi(s, t)$, s, $t\epsilon Q$, then $f(\phi)\epsilon R$. (B) In order that $\phi\epsilon R$, it is sufficient that it be true locally, that is, that to any s_0, t_0 there exist a $\phi_0\epsilon R$ which coincides with ϕ over a neighborhood of s_0, t_0. *F. Bohnenblust.*

Krein, Mark. A ring of functions on a topological group. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 275–280 (1940). [MF 3888]

A complex valued function φ defined over a group G is Hermitian or positive definite if the kernel $\varphi(gh^{-1})$, g, $h\epsilon G$, is Hermitian or positive definite, respectively. Assuming the group G to be either compact or commutative the author proves the theorem: If a continuous Hermitian function φ can be represented as the difference between two positive definite functions, then it can be represented as the difference between two continuous positive definite functions. *F. Bohnenblust* (Princeton, N. J.).

Krein, Mark. On almost periodic functions on a topological group. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 5–8 (1941). [MF 4257]

The author defines for almost periodic (a.p.) functions on a topological group a generalization of the notion of an absolutely convergent trigonometric sum, and characterizes it in terms of positive definite quadratic forms. An a.p. function $\varphi(s)$ on a topological group G is said to belong to the ring R if it can be expressed as a finite linear combination $a_1\varphi_1(s)+\cdots+a_n\varphi_n(s)$ of a.p. functions $\varphi_1(s), \cdots, \varphi_n(s)$ such that $\varphi_j(st^{-1})$ $(s, t\epsilon G)$ is a continuous positive definite kernel $(j=1, \cdots, n$, and a_j is an arbitrary complex number). The author shows that, if $\varphi(s)\epsilon R$ and $a\epsilon G$, then $\varphi(as)$ and $\varphi(sa)\epsilon R$. Moreover he generalizes the theorem of Wiener on reciprocals of absolutely convergent Fourier series by showing that if $\varphi(s)\epsilon R$, then $f(\varphi(s))\epsilon R$, where $f(z)$ is an arbitrary analytic function which is regular on the closure of the set of values $z=\varphi(s)$ $(s\epsilon G)$. Finally, he gives an abstract formulation of Wiener's method of proof of his reciprocal theorem by showing that, if G is compact, then a necessary and sufficient condition that $\varphi(s)\epsilon R$ is that it should locally belong to R. *R. H. Cameron.*

Krein, Mark. On positive functionals on almost periodic functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 9–12 (1941). [MF 4258]

This paper is based on the preceding, and begins by giving a norm $\|\varphi\|$ for elements $\varphi(s)\epsilon R$. This norm is the sum of the absolute values of the generalized Fourier coefficients if $\varphi(s)$ is Hermitian, but is not quite this sum in the general case, which is made to depend on the Hermitian case. The ring R is complete by this norm $\|\varphi\|$, which agrees with the norm given earlier by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 355–359 (1940)] for linear combinations of positive definite functions.

Now let Γ_G be the linear envelope of the elements of all irreducible continuous representations of G. Then the theorem is proved that, if $F(\varphi)$ is a homogeneous additive functional on Γ_G such that $F(|\varphi|^2)\geqq 0$ for all $\varphi\epsilon\Gamma_G$, it follows that $F(\theta)\geqq 0$ for all real non-negative $\theta\epsilon\Gamma_G$. Finally, a new proof is given for a theorem of T. Tannaka [Tôkohu Math. J. **45**, 1–12 (1938)] on groups of elementary functionals. *R. H. Cameron* (Cambridge, Mass.).

Krein, M. Sur une généralisation du théorème de Plancherel au cas des intégrales de Fourier sur les groupes topologiques commutatifs. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 484–488 (1941). [MF 4464]

Let $G=\{g\}$ be a locally compact separable commutative group and let $X=\{\chi\}$ be its character group in the sense of Pontrjagin ($(g(\chi)=\chi(g)$ is a real numer modulo 1). Let $L_2(G)[L_2(X)]$ be the Hilbert space defined on $G[X]$ with respect to the Haar measure of $G[X]$. Then $f(g)\epsilon L_2(G)$ implies

$$P(f)=\varphi(\chi)=\text{l.i.m.}\int f(g)\exp(2\pi i\chi(g))dg\epsilon L_2(X),$$

and, conversely, $\varphi(\chi)\epsilon L_2(X)$ implies

$$Q(\varphi)=f(g)=\text{l.i.m.}\int\varphi(\chi)\exp(-2\pi ig(\chi))d\chi\epsilon L_2(G);$$

moreover, under suitable normalizations of Haar measures, P and Q are both unitary and $P\cdot Q=Q\cdot P=1$. If G (and

hence X) is the group of all real numbers with ordinary topology, then this is a theorem of Plancherel; if G is compact or discrete (and hence X is discrete or compact), then this is equivalent to the theorems of Peter-Weyl and Riesz-Fischer. The same result was obtained by A. Weil [L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 863, Paris, 1940] by using a theorem of L. Pontrjagin that such a group G has a discrete subgroup N such that the factor group G/N is compact. A. Weil proved that, if the theorem is true for N and G/N, then it is also true for G. The present proof does not use any result of the structure of such groups, but is based on the theory of normed rings and positive definite functions defined on groups developed by I. Gelfand, D. Raikov and the author.
S. Kakutani (Princeton, N. J.).

Raikov, D. Generalized duality theorem for commutative groups with an invariant measure. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 589–591 (1941). [MF 4466]

Using the result of the paper reviewed above, a new proof is given for the duality theorem of L. Pontrjagin for locally compact commutative groups. More generally, if $G=\{g\}$ is a commutative topological group with a non-trivial regular invariant measure $\mu(E)$ such that $\mu(E\cap E+g)$ is continuous in g for any fixed E with $\mu(E)<\infty$, then G can be embedded isomorphically in a locally compact commutative group G^* in such a way that G^*-G is of inner measure zero (with respect to the Haar measure of G^*) and such that the measure $\mu(E)$ is induced naturally by the Haar measure of G^*; G coincides with G^* whenever G is a measurable subset of G^*. The same result was obtained by A. Weil [C. R. Acad. Sci. Paris **202**, 1147–1149 (1936); L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 863, Paris, 1940] and K. Kodaira [cf. the following review] for general non-commutative groups and for one-sided invariant measures without assuming any topology on G. In this case, the regularity and the continuity of $\mu(E)$ should be replaced by a condition of "measurability" of $\mu(E)$ which assures the possibility of applying Fubini's theorem.
S. Kakutani (Princeton, N. J.).

Kodaira, Kunihiko. Über die Gruppe der messbaren Abbildungen. Proc. Imp. Acad. Tokyo **17**, 18–23 (1941). [MF 4334]

Kodaira, Kunihiko. Über die Beziehung zwischen den Massen und den Topologien in einer Gruppe. Proc. Phys.-Math. Soc. Japan (3) **23**, 67–119 (1941). [MF 4237]

As is well known, on every locally bicompact group, there exists a left invariant Haar measure which is uniquely determined up to a constant factor. The main purpose of these two papers (of which the first one is an abstract of the second) is to prove that, in some sense, the converse is true. Let G be a group (with or without topology) with a left invariant Carathéodory outer measure $m^*(A)$ such that $f(y^{-1}x)$ is measurable whenever $f(x)$ is measurable (the measure on $G\times G$ being defined multiplicatively as usual). Then the neighborhood system of each point $x\epsilon G$ given by $\{xAA^{-1}\}$, where A runs over all measurable sets of positive measure, will determine a uniform topology T_{m^*} on G. If G is locally bicompact and if $m^*(A)$ is the left invariant Haar measure of G, then T_{m^*} is equivalent to the original topology of G. But, in general, G is not locally bicompact with respect to T_{m^*}; even the separation axiom may not be satisfied by T_{m^*}. In this case, let N be the totality of all $x\epsilon G$ which are not separated from the unit element e of G. Then N is an invariant subgroup of G, and the factor group G/N is locally totally bounded with respect to the topology of G/N induced naturally by T_{m^*}. Moreover, the locally bicompact group $\mathfrak{G}$, which is obtained from G/N by completion, contains G/N as a dense subgroup such that $\mathfrak{G}-G/N$ is of inner measure zero with respect to the left invariant Haar measure μ^* of $\mathfrak{G}$ (G/N coincides with $\mathfrak{G}$ whenever G/N is μ^*-measurable in $\mathfrak{G}$); and the measure $m_1^*(A)$ on G which is defined by $m_1^*(A)=\mu^*(r(A))$, where $r(x)$ is the natural mapping of G onto G/N, coincides with the original measure $m^*(A)$. As a corollary, a homomorphic mapping of a locally bicompact group onto another is continuous if and only if it is measurable with respect to their Haar measures. These results were previously announced in part by A. Weil [C. R. Acad. Sci. Paris **202**, 1147–1149 (1936), without proof], who discussed the case $N=(e)$. The general case when G is a group of measure preserving transformations of another space with measure is also treated in the first paper. The second paper contains, besides the proof of the main results, a detailed proof of the existence as well as a simpler proof of the uniqueness of Haar measure (in non-separable cases); the analogues of Vitali's covering theorem and of Lebesgue's density theorem are also discussed (in separable cases).
S. Kakutani (Princeton, N. J.).

Kantorovitch, L. Linear operations in semi-ordered spaces. I. Rec. Math. [Mat. Sbornik] N.S. **7** (49), 209–284 (1940). (English. Russian summary) [MF 2789]

The author considers linear semi-ordered spaces Y which satisfy the following axioms: (i) 0 is not >0, (ii) $y_1>0$, $y_2>0$ implies $y_1+y_2>0$; (iii) for every $y\epsilon Y$ there exists $y_1\epsilon Y$ such that $y_1>0$ and $y_1-y>0$; (iv) if λ is a positive real number and $y>0$ then $\lambda y>0$; (v) every set E bounded from above possesses an exact upper bound, sup E. Such spaces are denoted by K_s. If axiom (v) holds only for denumerable (finite) sets we have spaces $K_s^-(K_4)$, respectively. In a space K_s^- we define $\overline{\lim}\, y_n=\inf\sup(y_n, y_{n+1}, \cdots)$, $\underline{\lim}\, y_n=\sup\inf(y_n, y_{n+1}, \cdots)$. In case $\overline{\lim}\, y_n=\underline{\lim}\, y_n=y$ it is said that $y_n\to y(o)$. On the other hand it is said $y_n\to y(t)$ if every subsequence of $\{y_n\}$ contains another subsequence which $\to y(o)$. A space K_s is called regular (K_s^-) if, given a sequence of sets $E_n\subset X$ such that $\sup E_n\to y(o)$, then there is a sequence of finite subsets E_n' of E_n such that $\sup E_n'\to y(o)$. Various other spaces and modes of convergence are considered, but due to lack of space they can not be mentioned here.

An additive operation V on a space X to a space Y is said to be continuous (or linear) if convergence (in the sense t or o) $x_n\to x$ implies convergence (in the sense t or o) of $y_n=Ux_n$. Thus there appear four types of linear operations H_t^o, H_o^o, H_t^t, H_o^t. In the first chapter of the present paper the author considers various necessary and sufficient conditions that an additive operation would belong to one of these classes (and also to some other classes). As instances we mention the following results. If X and Y are of type K_s^{-1} then $H_o^t=H_t^t$, $H_t^o\subset H_o^o\subset H_t^t$. A linear operation is homogeneous. If an additive operation is continuous at a point (in some of the above senses) then it is continuous, in the same sense, in the whole space. A necessary and sufficient condition that an additive operation on a space $X\epsilon K_s^-$ to a space $Y\epsilon K_s^-$ should belong to the type H_t^t is that the image of every (t)-bounded set should be also (t)-bounded [a set E is called (t)-bounded if, for every $\{y_n\}\epsilon E$ and every sequence $\{\lambda_n\}$ of real numbers $\to 0$, we

have $\lambda_n y_n \to 0(t)$]. An additive operation (on a space of type K_4 to a space of the same type) is said to be positive if $x \geqq 0$ implies $Ux \geqq 0$. An additive operation U is said to be regular if there exists a positive operation U_1 such that $U_1 - U$ is positive. The class of all regular operations is denoted by H_r. If $X \epsilon K_4$, $Y \epsilon K_5$ then the set of all regular operations on X to Y constitutes itself a space of type K_5.

In Chapter II the author introduces various forms of abstract integrals and applies them to the integral representation of linear operations (in various senses) defined on various function spaces (such as l, l_p, L, L_p, C, M, etc.) to various abstract spaces. It is interesting to compare some of author's results with results of a recent paper by Dunford and Pettis [Trans. Amer. Math. Soc. 47, 323–392 (1940); these Rev. 1, 338]. *J. D. Tamarkin* (Providence, R. I.).

Stone, M. H. A general theory of spectra. II. Proc. Nat. Acad. Sci. U. S. A. 27, 83–87 (1941).

[The first part appeared in the same Proc. 26, 280–283 (1940); cf. these Rev. 1, 338.] This paper discusses the theory of lattice-ordered Abelian groups in relation to the investigations reported in part I. In particular, the author applies and generalizes F. Riesz's theory of operators, which does not use the Stieltjes integral [Ann. of Math. (2) 41, 174–206 (1940); these Rev. 1, 147]. An l-group L is defined by conditions which are equivalent to those on a lattice-ordered Abelian group. Since these conditions resemble the postulates for commutative rings, essential properties of l-groups can be obtained by arguments similar to those used in the ring-theory. Accordingly, the author states a series of conditions corresponding to those assumed for rings in part I; he reports some results concerning an extension of an l-group which satisfies some of those conditions to another l-group which fulfills more of them. For instance, the extension to an l-group which consists of the "formal fractionals" a/n, a being an element of L and n a natural number, is discussed; another example is the extension to an l-group which is "complete" in terms of a certain metric analogous to the metric treated in part I. Furthermore, the homomorphisms of an l-group are studied by means of their normal sub-l-groups which here play a role analogous to that of the ideals in ring-theory. Applying a principle of N. H. McCoy and D. Montgomery [see Duke Math. J. 3, 455–459 (1937)] the following theorems are obtained: (i) A complete l-group can be isomorphically imbedded as a sub-l-group in a certain complete l-group, with preservation of least upper and greatest lower bounds for sequences (in terms of lattice-theory). Now, using some arguments of the ring-theory and some topological complements, the author derives a function-theoretic representation, due to S. Kakutani [see Proc. Imp. Acad. Tokyo 16, 63–67 (1940); these Rev. 2, 69], which is analogous to the representation for rings in part I. (ii) An l-group L which satisfies some additional conditions is isomorphic to the l-group of all continuous real functions on a uniquely determined bicompact Hausdorff space $S(L)$; L is complete if and only if $S(L)$ is a Boolean space associated with a completely additive Boolean algebra. This yields the possibility of setting up in complete l-groups a calculus of operators as was done in part I, and thus of obtaining without use of integration results similar to those of F. Riesz. Further, by imbedding a complete l-group L in an l-group which satisfies some stronger requirements and by topological investigation, the author shows: (iii) There exists a unique bicompact Boolean space $S(L)$ representing a completely additive Boolean algebra, such that L is isomorphic to an l-group L_0 of certain continuous real functions on $S(L)$. Finally, he states in general what representations of different types of l-groups are possible, and indicates an extension of the parallel between the ring-representation in the first note and the l-group-representation in (iii). *E. D. Hellinger.*

Maeda, Fumitomo. Relative dimensionality in operator rings. J. Sci. Hirosima Univ. Ser. A. 11, 1–6 (1941). [MF 4461]

The author considers a ring M of operators on Hilbert space and, in this ring, the complemented non-modular lattice E of projections. He proves that the "center" of this lattice consists of the elements having unique complements. He gives three equivalent conditions for essential disjointness. Finally, he proves that either E is reducible (has a non-trivial center), or any two projections are comparable. The paper is closely related to the work of J. von Neumann and F. J. Murray on operator-rings. *G. Birkhoff.*

Phillips, R. S. On linear transformations. Trans. Amer. Math. Soc. 48, 516–541 (1940). [MF 3170]

This paper contains many theorems of representation for linear operators on a Banach space. Most of these representations are in terms of an integral with respect to a vector valued measure function. The following characterization of linear operators $U(\phi)$ on $L^p(0, 1)$ to an arbitrary Banach space X is typical. For every such operator there is an additive set function $x(\tau)$ defined for all measurable subsets τ of (0, 1) and with values in X such that

$$\text{l.u.b.} \sum_{\pi} |\bar{x}x(\tau_i)|^q / |\tau_i|^{q-1} = M_{\bar{x}} < \infty$$

(where the l.u.b. is taken over all partitions π of (0, 1) into measurable subsets τ_i) and such that $U(\phi) = \int \phi dx$. The norm $|U|$ is given by $\text{l.u.b.}_{|\bar{x}|=1} M_{\bar{x}}$ and the integral is defined as the π limit of the sums $\sum_\pi \phi(t_i) x(\tau_i)$, where $t_i \epsilon \tau_i$. The conditions of Cohen and Dunford for compactness in a space with a base are generalized to the case where there exists a partially ordered set U_π of completely continuous (c.c.) operators with $|U_\pi| \leqq M$ and $\lim_\pi U_\pi x = x$. This formulation contains the known results in L^p spaces. The representation theorem of Dunford and Pettis for weakly c.c. operators on L is generalized from the Euclidean to the abstract case. *N. Dunford* (New Haven, Conn.).

Kakutani, Shizuo. Concrete representation of abstract (L)-spaces and the mean ergodic theorem. Ann. of Math. (2) 42, 523–537 (1941). [MF 4299]

The author gives detailed proofs of the results announced in Proc. Imp. Acad. Tokyo 15, 121–123 (1939) [cf. these Rev. 1, 59]. He also proves that to any abstract (L)-space (AL) with a weak unit there corresponds a totally disconnected bicompact topological space Ω and a completely additive measure defined on a Borel field of subsets of Ω, such that (AL) is isometric and lattice-isomorphic to the Banach space of Lebesgue integrable functions on Ω.

G. Birkhoff (Cambridge, Mass.).

Riesz, Frédéric. Sur la théorie ergodique des espaces abstraits. Acta Univ. Szeged. Sect. Sci. Math. 10, 1–20 (1941). [MF 4427]

The author proves a mean ergodic theorem in abstract (L) spaces, that is, in Banach lattices (L) in which norm is additive on positive elements. [See G. Birkhoff, Proc. Nat. Acad. Sci. U. S. A. 24, 154–158 (1938).] Let T be a

bounded linear operation on (L) to (L) with $\|T\| \leqq 1$. If

$$-\bar{f} \leqq \varphi_n \equiv (1/n) \sum_{k=1}^{n} T^k(f) \leqq \bar{f}, \qquad n=1, 2, \cdots,$$

for some f, $\bar{f} \epsilon (L)$, then $1/(n-m)\sum_{k=m+1}^{n} T^k(f)$ converges strongly as $n-m \to \infty$. The proof is based on the fact that $\{\varphi_n\}$ contains a weakly convergent subsequence, or more generally, that $\{\varphi_n\}$ is Δ-convergent. (A sequence $\{\varphi_n\}$ is called Δ-convergent to φ if, for any $\epsilon > 0$ and for any r, there exists a convex combination $g_r = \sum_{k=r}^{s} c_k \varphi_k$, $c_k \geqq 0$, $\sum_{k=r}^{s} c_k = 1$, such that $\|\varphi - g_r\| < \epsilon$. $\{\varphi_n\}$ is Δ-convergent if it has a weakly convergent subsequence.) To prove this the author introduces a product operation xy in (L): let $l > 0$ be an arbitrary but fixed element of (L), and put $x^2 = \sup_{-\infty<\lambda<+\infty}(2\lambda x - \lambda^2 l)$. This exists if, for example, $-\lambda_0 l \leqq x \leqq \lambda_0 l$ for some $\lambda_0 > 0$. Now H is a linear subspace of (L); xy is then defined by $xy = \frac{1}{4}\{(x+y)^2 - (x-y)^2\}$. If we put $|||x||| = (\|x^2\|)^{\frac{1}{2}}$ and $(x, y) = \|xy \vee 0\| - \|xy \wedge 0\|$, then H is a real Hilbert space with these as its norm and inner product. If we now take $l = \bar{f}$, then all φ_n belong to H and $|||\varphi_n||| \leqq 1$. Hence $\{\varphi_n\}$ contains a weakly convergent subsequence (in H), and $\{\varphi_n\}$ is Δ-convergent in H. It is then easy to see that $\{\varphi_n\}$ is also Δ-convergent in (L). The same ergodic theorem is proved by the reviewer [Ann. of Math. (2) **42**, 523–537 (1941); see the preceding review] in another way, namely, by representing the given (L) as a concrete (L) space of integrable functions. It is possible to make l correspond to a function which is identically 1. Then the product xy introduced above is nothing but the product of two functions x, y in the ordinary sense, and H is exactly the (L^2) space with respect to this realization of (L). The ingenuity of the author's proof lies in the point that he avoids the concrete representation completely. The author also discusses the case when φ_n are almost bounded, that is, when there exists, for any $\epsilon > 0$, an $\alpha > 0$ such that $\|\varphi_n - \varphi_n \wedge \alpha\bar{f}\| < \epsilon$, $\|\varphi_n \vee (-\alpha\bar{f}) - \varphi_n\| < \epsilon$, $n = 1, 2, \cdots$. This clearly corresponds to the case of equi-integrable functions which was discussed previously by the author [Proc. London Math. Soc. (2) **13**, 274–278 (1938)]. *S. Kakutani* (Princeton, N. J.).

Riesz, Frederick. Another proof of the mean ergodic theorem. Acta Univ. Szeged. Sect. Sci. Math. **10**, 75–76 (1941). [MF 4434]

A simple proof is given for the mean ergodic theorem in Hilbert space H: If T is a bounded linear operation on H to H with $\|T\| \leqq 1$, then, for every $f \epsilon H$, $1/(n-m)\sum_{m+1}^{n} T^k(f)$ converges strongly as $n-m \to \infty$. The method of proof is quite similar to the one used by G. Birkhoff [Duke Math. J. **5**, 19–20 (1939)]. In both papers uniform convexity is used instead of weak compactness of the unit sphere.

S. Kakutani (Princeton, N. J.).

Wiener, Norbert and Wintner, Aurel. Harmonic analysis and ergodic theory. Amer. J. Math. **63**, 415–426 (1941). [MF 4161]

Let the space S of points P possess a measure and let τ be a metrically transitive measure preserving flow in S. Then for a Lebesgue integrable function $f(P)$ on S the limit

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{+n} e^{ij\lambda} f(\tau^j P)$$

exists for $-\infty < \lambda < \infty$ and for almost all P, where the excluded set of measure zero is independent of λ. This generalizes the ergodic theorem of G. D. Birkhoff in that the exceptional set is independent of λ. As an application the same type of generalization is given to the Kolmogoroff law of large numbers. *N. Dunford* (New Haven, Conn.).

Oldenburger, Rufus. Recurrence of symbolic elements in dynamics. Bull. Amer. Math. Soc. **47**, 294–297 (1941). [MF 4180]

Let T be a symbolic trajectory, that is, T admits a representation $\cdots a_{-1}a_0a_1a_2\cdots$, where the symbols a_i are chosen from a given class. The ray of pairs (a_{i-1}, a_i), (a_{i-2}, a_{i+1}), $\cdots$ is termed an element E based on T. The symbols which comprise E are pairs and two pairs are alike if and only if corresponding symbols in the pairs are alike. With this understood, recurrence of E is defined in the usual sense. The principal results of the paper are the two following theorems: (1) if the trajectory T is recurrent and identical with its inverse, each element based on T is recurrent; (2) if for some block C and ray R, T is of the form $R^{-1}CR$ and if each element based on T is recurrent, then T is identical with its inverse. *G. A. Hedlund.*

TOPOLOGY

Rosenthal, Artur. Verallgemeinerungen des Raumbegriffes. Chr. Huygens **18**, 234–250 (1940). [MF 4358]

Ribeiro, Hugo. La cohérence d'un ensemble et les ensembles denses en soi. Portugaliae Math. **2**, 67–76 (1941). [MF 4074]

From the point of view of general topology the author studies the coherence $c(X)$ of a set X $(c(X) = XX')$ for $X \subset [1, ']$ and the family of sets dense in themselves and shows how these notions may be used to furnish characterizations of topological spaces. *J. F. Randolph.*

Ribeiro, Hugo. Caractérisations des espaces réguliers normaux et complètement normaux au moyen de l'opération de dérivation. Portugaliae Math. **2**, 1–7 (1940). [MF 4073]

A neighborhood space 1 is said to be accessible if for the operation of derivation (I') $(A+B)' = A' + B'$, (II') $A'' \subset A'$, (III') $A' = 0$ if A consists of a single point. An accessible space is further designated as Hausdorff, regular, normal, completely normal if, respectively, (T_1) for a and b two points, there exist two disjoint sets to which a and b are respectively interior; (T_2) for a point a and a closed set F not containing a, there exist two disjoint sets to which a and F are respectively interior; (T_3) for E and F disjoint closed sets, there exist two disjoint sets to which E and F are respectively interior; (T_4) for G and H disjoint and not connected, there exist two disjoint sets to which G and H are respectively interior.

Fréchet proposed the problem of characterizing these spaces by the operation of derivation. Monteiro [Portugaliae Math. **1**, 333–339 (1940); cf. these Rev. **2**, 69] solved the problem for a Hausdorff space by showing in an accessible space the equivalence of (T_1) and the statement: If distinct points a_1 and a_2 belong to A', then A may be decomposed into two non-empty disjoint sets A_1 and A_2

such that A_1' does not contain a_2 and A_2' does not contain a_1. The present paper completes the solution of the problem by showing the equivalence of (T_2), (T_3), (T_4) and (D_1), (D_2), (D_3), respectively, where, with b a point and $B_1 \neq 0$, $B_2 \neq 0$ sets belonging to A', there exists a decomposition of A into two non-empty disjoint sets A_1 and A_2 such that (D_1) if $b \epsilon 1-B$, $b \epsilon 1-B_1'$, then $A_1'(b)=A_2'B=0$; (D_2) if $B_1B_2+B_1B_2'+B_2B_1'+B_1'B_2'=0$, then $A_1'B_2=B_2'B_1=0$; (D_3) if $B_1B_2+B_1B_2'+B_2B_1'=0$, then $A_1'B_2=A_2'B_1=0$. Also the author points out that only in showing the equivalence of (T_3) and (D_3) are all of (I′), (II′) and (III′) used. *J. F. Randolph.*

Rey Pastor, J. Spaces D_0. Univ. Nac. Tucumán. Revista A. **1**, 105–122 (1940). (Spanish) [MF 4058]

A space D_0 is one in which there is a distance function (x, y) such that $(x, y) \geqq 0$, $(x, x)=0$ and $(x, z) \leqq (x, y)+(y, z)$; it differs from a metric space in that neither the relation $(x, y)=(y, x)$ nor the condition $x=y$ if $(x, y)=0$ necessarily holds. The author gives some simple properties of convergence, closure and distance between points and sets in such spaces. He shows how spaces D_0 occur naturally as spaces whose elements are sets in other spaces; in particular, the subsets of a space D_0 can be made into another space D_0. Various kinds of limit in such spaces of subsets are discussed. The author calls attention to related work of Wilson [Amer. J. Math. **53**, 675–684 (1931)] and Kakutani [Proc. Phys.-Math. Soc. Japan (3) **18**, 641–658 (1936)].
R. P. Boas, Jr. (Durham, N. C.).

Albert, George E. A note on quasi-metric spaces. Bull. Amer. Math. Soc. **47**, 479–482 (1941). [MF 4541]

This paper continues a study of asymmetric distance functions begun by W. A. Wilson [Amer. J. Math. **53**, 675–684 (1931)]. Wilson's Axiom I is replaced by this weaker axiom: $xy=yx=0$ if and only if $x=y$; the triangle axiom is also assumed. Much of Wilson's paper remains valid with certain obvious changes; the present paper chiefly sketches new properties which arise under the weaker axioms. As in Wilson's paper, there are three types of limit point and accordingly three types of open and closed sets. The author obtains relationships between the three types of limit point and the relationship of being adjacent. The point x is said to be adjacent to y, if $xy=0\neq yx$. Note that this relationship is asymmetric. With a proper choice of neighborhood and of distance function, it is shown that a quasi-metric space satisfies Hausdorff's axioms A, B, C, D and Kolmogoroff's separation axiom, and that the converse is true if D is replaced by E. *H. M. Gehman.*

Alexits, Georges. Les espaces réguliers et le problème de métrisation. Comment. Math. Helv. **13**, 1–5 (1940). [MF 3984]

The author asserts that a countably compact Hausdorff space which satisfies the first axiom of countability is metrizable and hence (bi-)compact. His proof fails at the top of page 3. His theorem is disproved by the example of the ordinal numbers of the second number class with the topology induced by their ordering. *J. W. Tukey.*

Fomin, S. Erweiterungen topologischer Räume. Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 285–294 (1940). (Russian. German summary) [MF 3465]

The author is concerned with some topological results of Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)]. He wishes to obtain them by topological methods, rather than by the theory of Boolean rings. If R is a Hausdorff space, and $\mathfrak{G}=\{G\}$ is a basis for its open sets, then an end α of the field $\mathfrak{A}$ over $\mathfrak{G}$ is a maximal collection of sets of $\mathfrak{A}$ such that every finite number meet. Each set $A \epsilon \mathfrak{A}$ determines the set U_A of ends which contain A. These sets form a basis for the open sets of the space $\theta(\mathfrak{G}; R)$ of all ends of $\mathfrak{A}$. It is shown that $\theta(\mathfrak{G}; R)$ is a (bi-)compact Hausdorff and 0-dimensional space. This procedure is reminiscent of that used by Wallman [Ann. of Math. (2) **39**, 112–126 (1938)] for other purposes, but its success here depends on its modifications. Certain closed subsets of $\theta(\mathfrak{G}; R)$ are regarded as points of a topological space $X(R)$, which is shown to be isomorphic to R. The space whose points are these closed subsets and certain points of $\theta(\mathfrak{G}; R)$ is an H-closed space in which $X(R)$ is dense. The author proceeds to show that a Hausdorff space is (bi-)compact if and only if each of its closed subsets is H-closed. [Cf. Stone, l.c., and Katětov, Časopis Pěst. Nat. Fys. **69**, 36–49 (1940); these Rev. **1**, 317.]
J. W. Tukey (Princeton, N. J.).

Morita, Kiiti. On uniform spaces and the dimension of compact spaces. Proc. Phys.-Math. Soc. Japan (3) **22**, 969–977 (1940). [MF 3795]

The author defines the dimension of a normal space in terms of finite open coverings in a way equivalent to the usual one. The equivalence follows from the fact that all finite coverings of a normal space give rise to a suitable notion of uniformity. This fact is proved by the author independently of the reviewer [Convergence and Uniformity in Topology, 1940; these Rev. **2**, 67]. He proves that, if S is normal and if $\beta(S)$ is the (bi-)compaction of Čech, then S and $\beta(S)$ have the same dimension. This result also follows from results of Wallman [Annals of Math. (2) **39**, 112–126 (1938)] and P. S. Alexandroff [Rec. Math. [Mat. Sbornik] N.S. **5 (47)**, 403–423 (1939); these Rev. **1**, 318]. He examines the connection between the dimension of a (bi-)compact Hausdorff space and the existence of essential mappings of the space on finite dimensional complexes. He obtains results similar to those obtained independently by P. S. Alexandroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 619–622 (1940); these Rev. **2**, 177]. He then shows that these results can be extended to the case where S is normal and countably compact. Throughout the author uses the ideas and notation of André Weil's theory of uniform spaces, working in the (bi-)compact case where the uniformity is uniquely determined by the topology.
J. W. Tukey (Princeton, N. J.).

Fox, Ralph H. On the Lusternik-Schnirelmann category. Ann. of Math. (2) **42**, 333–370 (1941). [MF 4289]

An extensive study of the category of subsets of a space is made, and the relation to homotopy and homology properties is considered. A set $A \subset M$ is here called categorical if some neighborhood U of it is contractible in M. The category $\text{cat}_M X$ of X in M is the minimal number of sets in a categorical covering of X. Properties of the category for sums of sets, etc., are given. A finite sequence $\{A_1, A_2, \cdots, A_k=X\}$ is categorical if the A_i are closed, $A_i \subset A_{i+1}$, and $A_1, A_2-A_1, \cdots, A_k-A_{k-1}$ are categorical. It is shown essentially that $\text{cat}_M X$ is the minimum length k of categorical sequences; one may choose $\dim A_i < \dim A_{i+1}$ if $\dim X$ is finite. Hence (Lusternik-Schnirelmann-Borsuk) $\text{cat}_M X \leqq 1+\dim X$. A similar inequality in terms of the number of dimensions for which there exist "essential" subsets of X is given. A characterization of category as a numerical set function of X is made. We shall touch briefly on the rest of the paper. The n-dimensional homotopy and homotopy

categories $h_n$cat and $H_n$cat of $X \subset M$ are defined as category was, with categorical coverings replaced by coverings by sets A as follows: Every continuous complex in A can be deformed to a point in M; or, every k-cycle ($k \leqq n$) in A bounds in M. If M is an n-dimensional absolute neighborhood retract, then $h_n\mathrm{cat}_M X = \mathrm{cat}_M X$ for closed X. If in the orientable n-manifold M there are cycles $\gamma^{r_1}, \cdots, \gamma^{r_k}$ ($n-r \leqq r_i \leqq n-1$) whose intersection cycle is not ~ 0, then $H_n$cat $M \geqq k+1$, with proper coefficient groups. Using P_n = projective n-space, and integers mod 2, $H_1\mathrm{cat}_{P_n} P_n = \mathrm{cat}_{P_n} P_n = n+1$. Some other topics considered are deformations, product spaces, covering spaces, identifications, the fundamental group, spaces with small categories and "strong categories." *H. Whitney* (Cambridge, Mass.).

Chow, Shao-Lien. Sur les ensembles fermés punctiformes. J. Chinese Math. Soc. 2, 235–237 (1940). [MF 4210]

A set is called punctiform (punctiforme, diskontinuierlich) provided it contains no continua. The author proves that any two points not belonging to a given closed punctiform subset E of n-dimensional Euclidean space may be joined by a simple arc of finite length that contains no points of E. Since the intersection of E with a plane is closed and punctiform, the theorem is an obvious consequence of the validity of its analogue in the plane which the author demonstrates with the help of a lemma based upon notions and results due to G. Bouligand [cf. Introduction à la géométrie infinitésimale directe, Paris, 1932, pp. 34–41]. *L. M. Blumenthal* (Columbia, Mo.).

Tscherkassoff, A. Sur les suites d'ensembles également homéomorphes. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 349–361 (1940). (Russian. French summary) [MF 3468]

The author considers a family of homeomorphic mappings $\{X_\nu\}$ of a compact (metric) space F_0 into another compact space. He calls the collection of sets $F_\nu = X_\nu(F_0)$ a system of uniformly homeomorphic sets if for every positive number ϵ a positive number η can be found such that $\rho(\alpha_1, \alpha_2) < \eta$ ($\alpha_1 \epsilon F_0$, $\alpha_2 \epsilon F_0$) implies $\rho(X_\nu(\alpha_1), X_\nu(\alpha_2)) < \epsilon$ and $\rho(X_\nu(\alpha_1), X_\nu(\alpha_2)) < \eta$ implies $\rho(\alpha_1, \alpha_2) < \epsilon$. Main result: If a sequence of uniformly homeomorphic sets $\{F_n\}$ converges to a set F, then F is homeomorphic to the sets F_n. Furthermore, the author investigates families of uniformly homeomorphic simple arcs. *W. Hurewicz* (Chapel Hill, N. C.).

Kolmogoroff, A. Points of local topologicity of enumerably folded open mappings of compacts. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 479–481 (1941). [MF 4462]

A mapping $f(X) = Y$ of a metric compact space X is open if for every open set U in X the subset $f(U)$ of Y is open; f is enumerably folded if the set $f^{-1}(y)$ is at most countable for every $y \epsilon Y$. A point $x \epsilon X$ is a point of local topologicity of f if there is a neighborhood U of x on which f is 1-1. The author proves that if f is open and enumerably folded then the points of local topologicity of f are dense in X. *S. Eilenberg* (Ann Arbor, Mich.).

Hopf, Heinz. Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume. Vierteljschr. Naturforsch. Ges. Zürich 85 Beiblatt (Festschrift Rudolf Fueter), 165–177 (1940). [MF 4415]

From a problem in algebra concerning systems of real symmetric bilinear forms to a question regarding topological imbedding of one manifold within another might seem a far cry, although perhaps not to one familiar with recent intimacies between algebra and topology. The algebraic problem here proposed [see E. Stiefel, Comment. Math. Helv. 8, 349 (1936) and F. Behrend, Compositio Math. 7, 1–19 (1939); cf. these Rev. 1, 36] is to determine, for each fixed natural number $r > 1$, a number $N(r)$ defined as follows: Let $f^i(x, y) = 0$, $i = 1, 2, \cdots, n$, be a system of equations in which f^i is a real symmetric bilinear form $\sum a^i_{jk} x_j y_k$, $a^i_{jk} = a^i_{kj}$, in the variables $x_1, \cdots, x_r$ and $y_1, \cdots, y_r$, and for which the only real solutions $(x_1, \cdots, x_r, y_1, \cdots, y_r)$ are those with $x_1 = \cdots = x_r = 0$ and those with $y_1 = \cdots = y_r = 0$; then the system of functions f^i is called definite. The number $N(r)$ is defined as the smallest number n for which there exists a definite system of n forms f^i. Thus $N(2) = 2[f^1 = x_1y_1 - x_2y_2$, $f^2 = x_1y_2 + x_2y_1]$.

That $N(r) \leqq 2r-1$, and for r even $N(r) \leqq 2r-2$, follows from simple algebraic considerations. To determine a lower bound for $N(r)$ the author reasons as follows: Consider $x = (x_1, \cdots, x_r)$, $y = (y_1, \cdots, y_r)$ as points of $(r-1)$-dimensional projective space P^{r-1}. Let $z_1, \cdots, z_n$ be coordinates for the Euclidean n-space E^n and let S^{n-1} denote the sphere of radius 1 and center the origin in E^n. If the system $f^1, \cdots, f^n$ is definite, then for no point x of P^{r-1} do all forms $f^i(x, x)$ vanish. Hence the functions $z_i(x) = f^i(x, x) / \sum_m [f^m(x, x)]^2$ are continuous and generate a single-valued mapping of P^{r-1} into S^{n-1}; moreover the mapping is shown to be one-to-one and hence a homeomorphism. Thus to every definite system of n symmetric bilinear forms in twice r variables corresponds a topological mapping of the projective space P^{r-1} into the sphere S^{n-1}; in particular, P^{r-1} may be imbedded topologically in $S^{N(r)-1}$. Since $N(r) \geqq r$ and P^{r-1} is imbeddable in S^{r-1} only for $r-1=1$, the case $N(r) = r$ can occur only for $r=2$. Then for $r>2$, $N(r) > r$, and the mapping of P^{r-1} into $S^{N(r)-1}$ may be considered a mapping into $E^{N(r)-1}$. Thus we arrive at the topological question: Into what Euclidean spaces may P^k be topologically mapped? Using the Gordon [Ann. of Math. (2) 37, 519–525 (1936)] extension of the Alexander duality theorem, the author shows that if a k-dimensional closed manifold is imbeddable in E^{k+1} then the additive group of its positive dimensional homology classes is the direct sum of two rings, from which it follows that, for $k>1$, P^k is not imbeddable in E^{k+1}. Thus $N(r) \geqq r+2$, and in particular $N(3) = 5$, $N(4) = 6$.

Byproducts: From the fact that $N(r) = r$ only for $r=2$ follows that even without postulating the associative law of multiplication, the field of complex numbers is the single commutative extension field of finite degree over the real number field. And from $N(3) = 5$, for example, follows that in the projective plane there exists for every four real conics at least one real pair of points conjugate to all four curves. *R. L. Wilder* (Ann Arbor, Mich.).

Nöbeling, Georg. Geometrische (Realitäts-)Ordnung und topologische Struktur. J. Reine Angew. Math. 183, 37–67 (1940). [MF 4227]

Given a system S of arbitrary sets C in a metric space R; let A be another metric space and f be a single-valued continuous transformation of A into R. If the number of points of A the images of which lie in a set C is finite or bounded or has the maximum s for all C, the order of A with respect to S for the imbedding f is said to be respectively finite, bounded or s. If not only A itself has the order s with respect to S for f, but if the same holds for arbitrarily small neighborhoods of every point of A, A is said to have the totally homogeneous order s with respect to S for f. The author studies the conditions under which a given compact

metric space A of finite dimension n has a finite or bounded or totally homogeneous order with respect to the set S of all m-dimensional linear manifolds in Euclidean k-space $R=R_k$ for a conveniently chosen imbedding f. Of the many results of this paper, only the following may be stated: If $m+n<k$, then there are, for any integer $s>(m+1)(k-m)\times(k-m-n)^{-1}-1$, imbeddings f for which the order of A with respect to S is not greater than s. (The case $m=0$, $k\geqq 2n+1$, $s=1$ gives the imbedding theorem of dimension theory.) If, furthermore, A is dense in itself, there are imbeddings f of A into R_k for which the order of A with respect to S is totally homogeneous and equal to s. If $m+n=k$, a sufficient condition for A having a bounded order with respect to S for a conveniently chosen imbedding f is that A is homeomorphic to an n-dimensional polyhedron of a Cartesian space. If $m+n>k$, the order of A with respect to S is finite for no f. Besides the set of all m-spaces in R_k, certain subsets of it are considered. In one section, the analogous problem for the order defined by means of the dimension instead of the number is solved.

P. Scherk (New Haven, Conn.).

Toranzos, Fausto I. On the singularities of Jordan curves. Union Mat. Argentina, Publ. no. 13, 8 pp. (1939). (Spanish) [MF 3853]

The author studies simple Jordan curves. For $P_0=P(t_0)$ fixed, let $\omega(t)$ be the argument of the secant joining P_0 to $P=P(t)$. Then P_0 is a spiral point if either (a) $\omega(t_0-0)$ is infinite, or (b) the discontinuity of $\omega(t)$ on the left at t_0 is more than 2π. It is shown that there is always a dense set of t's corresponding to non-spiral points, and an example is given where a dense set of t's correspond to spiral points.

J. W. Tukey (Princeton, N. J.).

Whitney, Hassler. On regular families of curves. Bull. Amer. Math. Soc. 47, 145–147 (1941). [MF 3831]

A family of curves filling a metric space is called regular if, given a point p of the space, there exists a neighborhood of p in which the family is homeomorphic with a family of straight lines. The author here gives a sufficient condition for a family to be regular; from this condition a restriction present in earlier sufficient conditions is omitted. The result is that regularity is guaranteed if, given any point p and a direction on the curve through p, there is an arc pq in this direction with the property that for every $\epsilon>0$ there is a $\delta>0$ such that, for any point p' within a distance δ of p, there is an arc $p'q'$ of the curve through p' which lies within an ϵ-neighborhood of pq and on which q' lies within an ϵ-neighborhood of q. In an earlier sufficient condition, the author had imposed on $p'q'$ the additional restriction that, if r' and s' are two points on $p'q'$ within a distance δ of each other, then the diameter of the arc $r's'$ is less than ϵ. A misprint occurs in the third line above equation (8), where the symbol s should be s'; also in the third line below equation (7) the word "furthest" does not refer to the metric of the space, but means something like "furthest along the arc pq."

C. B. Tompkins (Princeton, N. J.).

Kaplan, Wilfred. Regular curve-families filling the plane, I. Duke Math. J. 7, 154–185 (1940). [MF 3396]

A family of curves filling a region is called regular if near any point it is homeomorphic with a family of straight lines. Regular families F filling the plane R are studied. Some classical theorems of Bendixon [Acta Math. 24, 1–88 (1901)] on solutions of differential equations are proved with the solutions replaced by the curves of F. Next the relative positions of the curves of F are considered abstractly; of three curves C_1, C_2, C_3, one may lie between the other two, or each C_i may have the other two on a single side of it (that is, in a single component of $R-C_i$). Some obvious laws are assumed as postulates; for instance, C_2 between C_1 and C_3, and C_3 between C_2 and C_4, imply C_2 between C_1 and C_4. The curves now form the elements of a "chordal system." With its help, and the help of cross-sections in F, it is shown that F can be subdivided into (possibly countably many) subfamilies, each of the structure of the parallel lines of a half plane, the subfamilies fitting together in a definite manner. It is then proved that the curves of F may be considered as the level curves of a continuous function defined in R.

H. Whitney (Cambridge, Mass.).

Kaplan, Wilfred. Regular curve-families filling the plane, II. Duke Math. J. 8, 11–46 (1941). [MF 3944]

In the paper reviewed above the author showed that, in any regular family F of curves filling the plane, each curve goes to infinity in both directions, and that the relations between the curves is the same as the relation between the "chords" of an (abstractly defined) "normal chordal system" (NCS). It is shown here that every NCS may be represented as a set of non-intersecting chords of a circle S. Also, two families F_1 and F_2 are equivalent under a homeomorphism of the plane if and only if NCS_1 and NCS_2 are isomorphic. As the NCS may be classified in terms of the order of the end points of the chords on S, this gives a classification of the families F.

H. Whitney.

de Kerékjártó, Béla. Sur les fondements de la géométrie des cercles. Mat. Fiz. Lapok 47, 48–57 (1940). (Hungarian. French summary) [MF 2653]

Dans ce mémoire, j'expose les deux méthodes différentes qui servent à édifier la géométrie des cercles. La première, donnée par van der Waerden et Smid, consiste à caractériser le système des cercles situés sur la sphère. L'autre concerne à déterminer le caractère topologique du groupe homographique de la sphère; cette détermination est fournie par le théorème suivant dont la démonstration sera publiée dans un prochain mémoire de l'auteur: Soit G un groupe de transformations topologiques de la surface d'une sphère en elle-même conservant le sens, et satisfaisant aux conditions suivantes: Pour deux triples de points A, B, C et A', B', C', il existe une transformation de G et une seule qui transforme A, B, C en A', B', C'; celle-ci varie continuement avec le triple A', B', C'. Sous ces conditions, le groupe G est homéomorphe au groupe homographique de la sphère.

Author's summary.

de Kerékjártó, B. Sur les groupes transitifs de la droite. Acta Univ. Szeged. Sect. Sci. Math. 10, 21–35 (1941). [MF 4428]

It is shown that every simple transitive group of homeomorphisms of a circle with itself is continuous; so is every triply transitive group. Other degrees of transitivity can not exist. The former group is equivalent topologically to the rotations, the latter to the linear fractional transformations of one real variable. These characterizations, as well as those of transitive groups on a line, also obtained here, are due to Brouwer. A simplifying element in the present proofs is the special part played by those transformations which are of period two.

P. A. Smith (New York, N. Y.).

de Kerékjártó, B. Sur le groupe des homographies et des antihomographies d'une variable complexe. Comment. Math. Helv. 13, 68–82 (1940). [MF 3990]

Denote by L the group of all the linear fractional trans-

formations of the complex plane with non-vanishing determinant; and denote by L' the group generated by adjoining to L the transformation: $z'=\bar{z}$. A continuous group T of topological transformations of the two-dimensional sphere S is essentially the same as L if, and only if, there exists for every pair of triplets of points in S one and only one transformation in T mapping the first upon the second; and T is essentially the same as L' if, and only if, there exist for every pair of triplets of points in S exactly two transformations in T mapping the first triplet upon the second one. The author gives a purely geometrical proof of the second of these results without making use of the first one [a proof of which will be published in J. Math. Pures Appl.]. The proof proceeds in two steps: if a, b, c are three points on S, then a group T of the kind considered contains one and only one transformation not equal to 1 which leaves these points invariant; such a transformation is involutorial and its fixed-points are of the type of a circle. These "circles" make it possible to introduce a geometry on S. The subgroup of T consisting of those transformations which preserve orientation can then be shown to be essentially the same as L. *R. Baer* (Urbana, Ill.).

Morita, Kiiti. H. Hopf's extension theorem in normal spaces. Proc. Phys.-Math. Soc. Japan (3) **23**, 161–167 (1941). [MF 4518]

The principal result reads: The set of homotopy classes of maps of the n-dimensional bicompact or normal space S into the n-sphere is in 1-1 correspondence with the character group of the nth homology group mod 1 of S. This extension of the Hopf, Freudenthal, Bruschlinsky theorems is included in results announced by C. H. Dowker [Proc. Nat. Acad. Sci. U. S. A. **23**, 293–294 (1937)] but not yet published. The author's proof of the bicompact case is complete, while that of the normal case is merely sketched. The latter is not satisfactory since no distinction is made between uniform and non-uniform homotopies, a point which Dowker showed to be vital. *N. E. Steenrod* (Chicago, Ill.).

Hurewicz, W. and Steenrod, N. E. Homotopy relations in fibre spaces. Proc. Nat. Acad. Sci. U. S. A. **27**, 60–64 (1941).

A topological space X, metric space B and continuous mapping π of X onto B define a "fibre space" if for some $\epsilon_0>0$ a continuous function $\phi(x, b)$ exists, with values in X, defined whenever $\rho(\pi(x), b)<\epsilon_0$, and such that $\pi\phi(x, b)=b$ and $\phi(x, \pi(x))=x$. Thus ϕ maps the "fibres" $\pi^{-1}(b')$ into themselves for b' near b. The fibre bundles of the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 148–153 (1940); these Rev. **1**, 220] are fibre spaces. The homotopy groups $\pi_i(X)$, $\pi_i(B)$ and $\pi_i(X, F)$ (defined by mapping an i-cell into X so that its boundary and a fixed point of it go into F and a fixed point of it, respectively) are studied. It is shown that

$$\pi_i(X, \pi^{-1}(b_0))=\pi_i(B, b_0).$$

If $F\subset S^n$ (n-sphere) is closed and arcwise connected, then

$$\pi_i(S^n, F)=\pi_i(S^n)+\pi_{i-1}(F).$$

Among applications, we mention:

$$\pi_i(S^n)=\pi_i(S^{2n-1})+\pi_{i-1}(S^{n-1}), \qquad n=2, 4, 8.$$

The sphere of least dimension which can be a proper fibre space over S^n is S^{2n-1}; if the fibres are spheres, only S^{2n-1} can occur. *H. Whitney* (Cambridge, Mass.).

Whitehead, J. H. C. On adding relations to homotopy groups. Ann. of Math. (2) **42**, 409–428 (1941). [MF 4291]

Let X be an arcwise connected space and let $A_1, \cdots, A_k$ be disjoint closed n-cells such that XA_i is the boundary $(n-1)$-sphere $\dot{A}_i$ of A_i. This paper continues the study [begun in Proc. London Math. Soc. (2) **45**, 243–327 (1939)] of the relationships between the homotopy groups of X and $X^*=X+\sum_{i=1}^k A_i$. The principal tools are an operator α^ρ and a multiplication $\alpha\cdot\beta$ described below. For any element α of the rth homotopy group π_r [written multiplicatively for every r in this review] and element $\xi\epsilon\pi_1$ an element $\alpha^\xi\epsilon\pi_r$ [written $\xi\alpha$ if $r>1$ and π_r is written additively] is defined; $\alpha^\xi=\xi\alpha\xi^{-1}$ when $r=1$. Operators from the integral group ring $\mathfrak{R}$ over the fundamental group π_1 are defined by the formula

$$\alpha^{\sum_i c_i\xi_i}=\Pi_i(\alpha^{\xi_i})^{c_i}\epsilon\pi_r.$$

For any elements $\alpha\epsilon\pi_m$, $\beta\epsilon\pi_n$ an element $\alpha\cdot\beta\epsilon\pi_{m+n-1}$ is defined. This product is commutative when mn is even. If $m>1$, $n>1$ the product $\alpha\cdot\beta$ is a group multiplication; $\alpha\cdot\beta=\beta^{\alpha-1}$ when $m=1$. Generalized products using rotation groups are defined but not used. If S_1^m and S_2^n are spheres of dimension $m>1$ and $n>1$ which intersect in a single point, then $\pi_{m+n-1}(S_1^m+S_2^n)$ is the direct product of $\pi_{m+n-1}(S_1^m)$, $\pi_{m+n-1}(S_2^n)$ and the infinite cyclic group generated by $\alpha\cdot\beta$, where α and β generate $\pi_m(S_1^m)$ and $\pi_n(S_2^n)$, respectively. The proof of this theorem makes use of an interesting generalization of Hopf's invariant.

Let χ denote the natural homomorphism of $\pi_n(X)$ into $\pi_n(X^*)$. [N.B. On p. 418 from l. 17 to bottom replace ψ by χ.] If $n>2$ the kernel of χ is the smallest subgroup of $\pi_n(X)$ which is invariant under the group ring operators and contains all elements of the form $\alpha\cdot\beta$, where $\alpha\epsilon\pi_{n-j}(\dot{A}_i)$ and $\beta\epsilon\pi_{j+1}(X)$, $j=0, 1$. Let $\mathfrak{M}$ denote the Abelian group generated by elements e_i^ξ, where $i=1, \cdots, k$ and $\xi\epsilon\pi_1(X)$, subject to the relation $e_i^\xi e_i^\eta=e_i^{\xi+\eta}$. If $n=2$, let h_{π_1} denote the group generated by elements e_i^ξ subject to the relations $e_i^{\xi\alpha_i}=e_i^\xi$ and $e_i^\xi e_j^\eta=e_j^{\alpha_i^\xi\eta}e_i^\xi$, where α_i generates $\pi_1(\dot{A}_i)$. The formula $\phi(e_i^\xi)=\alpha_i^\xi$ defines a homomorphism of $\mathfrak{M}$ into $\pi_{n-1}(X)$ and for $n=2$ a homomorphism of h_{π_1} into $\pi_1(X)$. For $n>2$ a homomorphism ψ of $\pi_n(X^*)$ into $\mathfrak{M}$ and for $n=2$ a homomorphism ψ of $\pi_2(X^*)$ into h_{π_1} is defined such that, for any $\rho\epsilon\mathfrak{R}$, $\psi((\alpha^*)^\rho)=(\psi(\alpha^*))^\rho$. The kernel of ψ is $\chi(\pi_n(X))$ and $\psi(\pi_n(X^*))$ is the kernel of ϕ. If X^* is a complex and $n=2$, the kernel of χ is described in terms of the homotopy ring of X. The kernel of χ is also described when X is a 2-dimensional complex and $n=2$. Using the homomorphism ψ, a set of generators and relations for the second homotopy group of a 2-dimensional complex is exhibited. The unsolved problem: "Is any subcomplex of an aspherical 2-dimensional complex itself aspherical?" is briefly considered. *R. H. Fox* (Urbana, Ill.).

Alexandroff, P. Die Bettischen Gruppen und der Homologiering eines lokal-bikompakten Raumes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 623–626 (1940). [MF 3570]

This note defines homology and cohomology groups of a normal locally bicompact space in terms of the barycentric nerves of multiplicative coverings, and forms the associated cohomology ring. For a detailed account see the final part of the paper reviewed below. *A. W. Tucker.*

Alexandroff, Paul. General combinatorial topology. Trans. Amer. Math. Soc. **49**, 41–105 (1941). [MF 3667]

The results of this paper are parallel to those of the "dual"

theories of homology outlined by Alexander [Ann. of Math. (2) **37**, 698–708 (1936)] and Kolmogoroff [C. R. Acad. Sci. Paris **202**, 1144, 1325, 1558, 1641 (1936)]. However, the methods follow more standard lines: the author's projection spectra [Ann. of Math. (2) **30**, 101–187 (1929)], Pontrjagin's formulation of duality [Math. Ann. **105**, 165–205 (1931)] and Čech's cycles [Fund. Math. **19**, 149–183 (1932)]. A joint theory of homology and cohomology is developed systematically, first for ordinary simplicial complexes, then for directed (partially ordered) sets of complexes which map simplicially into their predecessors, and finally for normal spaces and open sets thereof. Following Steenrod [Amer. J. Math. **58**, 661–701 (1936)] the homology and cohomology groups of a normal space R are defined as the limit groups of the inverse and direct spectra formed from the corresponding groups of the nerves of finite open coverings of R. The definition of the groups of an open set in R takes special account of the covering elements whose closures are bicompact. The main result is of the order of the classical duality theorem of Alexander; it was first stated by Kolmogoroff [l.c., p. 1642]: if R is a locally bicompact space whose r- and $r+1$-dimensional cohomology groups are null, then the rth cohomology group of any closed set A in R is isomorphic to the $r+1$st group of $A-R$. The final part of the paper gives a second definition of homology and cohomology groups, based on the barycentric subdivisions of finite open coverings of the space. This definition, equivalent to the first, lends itself to the simple construction of a cohomology ring of the sort considered by Alexander [l.c.].

A. W. Tucker (Princeton, N. J.).

Alexandroff, P. S. General theory of homology. Uchenye Zapiski Moskov. Gos. Univ. Matematika **45**, 3–60 (1940). (Russian) [MF 3731]

Russian version of the paper reviewed above.

Verčenko, I. Sur les continus acycliques transformés en eux-mêmes d'une manière continue sans points invariants. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 295–306 (1940). (Russian. French summary) [MF 3466]

K. Borsuk [see Fund. Math. **24**, 51–58 (1935)] gave an example of a locally connected acyclic continuum which can be mapped continuously into itself without fixed points (this shows that Lefschetz's fixed point formula cannot be extended to arbitrary locally connected continua). The author constructs a 3-dimensional continuum (in Euclidean 4-space) which has the properties of Borsuk's example and in addition is simply connected. *W. Hurewicz.*

Smith, P. A. Transformations of finite period. III. Newman's theorem. Ann. of Math. (2) **42**, 446–458 (1941). [MF 4294]

This paper contains generalizations of that theorem of M. H. A. Newman which states that periodic transformations of given period p in a connected locally Euclidean space cannot be arbitrarily small [Quart. J. Math., Oxford Ser. **2**, 1–8 (1931)]. Let M be a Hausdorff space in which every open set is the sum of a countable family of closed sets, g a coefficient group for Čech cycles in M, $n\geqq 0$ an integer, T a 1-1 bicontinuous transformation of M into itself. M is said to have property p_n over g at a point d if there is a neighborhood $D(d)$ and an n-cycle Ω mod $M-D$ with coefficients in g (called, below, a relative cycle) such that: (1) $D'(d)\subset D$ implies Ω not ~ 0 mod $M-D'$; (2) if $B(d)\subset D$ there is a $B'(d)\subset B$ such that Ω is a basis for n-cycles mod $M-B$ relative to homologies mod $M-B'$; (3) cycles mod $M-B$ of dimension greater than n are ~ 0 mod $M-B'$. Such a pair (Ω, D) is called an n-fundamental pair for d over g; M has property q_n over g at d if for every neighborhood $A(d)$ of d there is an $A'(d)\subset A$ such that C a non-empty open subset of A' and Λ an n-cycle mod $M-A$ over g in $M-C$ imply $\Lambda\sim 0$ mod $M-A'$. If M has p_n, q_n for g at each point, M is called n-regular over g. Newman's theorem becomes: Theorem I. Let M be connected locally bicompact and finite dimensional, N an open set in M whose closure is bicompact, $q\geqq 2$ an integer, p a prime factor of q, g_p the additive group of integers reduced mod p. Then if M is n-regular over g_p there is a finite covering $\mathfrak{A}$ of M by open sets such that T of period q implies the existence of a point $x\subset N$ such that x and Tx are not in the same open set of $\mathfrak{A}$. If at each point d there is a D such that for every prime p there is an n-fundamental pair (Ω^p, D) relative to g_p such that the B' of (2), (3) may be found independent of p, and if in addition there is for given D' independent of p a finite covering U by open sets such that the coordinate $\Omega^p(U)$ of Ω^p on the nerve of U satisfies $\Omega^p(U)$ not ~ 0 mod $M-D'$ over g_p, then p_n is satisfied uniformly. Similarly q_n is satisfied uniformly if A' may be found independent of p. The orbit of $x\epsilon M$ is the set of all images of x under positive and negative powers of T. Theorem II: Let M be locally bicompact and finite dimensional, N an open set of M whose closure is bicompact. If properties p_n and q_n are satisfied uniformly in M there is a covering $\mathfrak{A}$ of M such that T periodic implies the existence of a point x of N such that the orbit of x is not contained in any $A\epsilon\mathfrak{A}$. The proof of I depends on the proof of II, which involves the technique developed in the author's earlier papers [Ann. of Math. (2) **39**, 127–164 (1938); **40**, 690–711 (1939); cf. these Rev. **1**, 30]. A side result, corresponding to another of Newman's theorems [loc. cit.], is that the set of fixed points in M of a periodic T is nowhere dense in M.

The author states that for M_n an ordinary n-sphere of radius r, the same methods will give the following sharper result, where a cap is any closed region on M smaller than a hemisphere and bounded by an $(n-1)$-sphere. Every periodic transformation operating in M has at least one orbit that is not contained in any cap. Furthermore, if T has period q and has fixed points, and d denotes geodesic distance, there is at least one point x such that $d(x, Tx) = 2\pi r/q$, and there will be a point y and integer s, $2\leqq s\leqq q$, such that s points of the orbit of y lie on the same great $(s-2)$-sphere of M_n. As an introduction to this paper the theorem just stated is proved by elementary methods in the case $n=2$. *W. W. Flexner* (Ithaca, N. Y.).

Martin, Venable and Roberts, J. H. Two-to-one transformations on 2-manifolds. Trans. Amer. Math. Soc. **49**, 1–17 (1941). [MF 3665]

J. H. Roberts has proved in a recent paper [Duke Math. J. **6**, 256–262 (1940); cf. these Rev. **1**, 319] that no continuous 2-to-1 transformation (that is, a continuous transformation for which the inverse image of every point consists of exactly two points) can be defined over a closed 2-cell. In the present paper the authors investigate the more general problem of existence of 2-to-1 transformations over compact 2-manifolds (with or without bounding curves). Not only the problem of existence of such mappings is solved, but in addition the collection of all image spaces is determined. A necessary and sufficient condition for the existence of a 2-to-1 continuous transformation defined over

a 2-manifold M with n bounding curves ($n=0, 1, 2, \cdots$) is that the Euler characteristic of M be an even number. A compact space B is an image of M under a 2-to-1 transformation if and only if B satisfies the following conditions: (1) B can be obtained by the identification by pairs of a finite number of interior points of a compact 2-manifold with k bounding curves, where $\frac{1}{2}n \leqq k \leqq n$. (2) The Euler characteristic of B is one-half of the Euler characteristic of M. In establishing these results an extensive use is made of the results and methods developed in Roberts' paper quoted above. The basic idea of both papers consists in associating with a given 2-to-1 transformation T defined over the manifold M the real valued function $f(x)$ = distance from the point $x \epsilon M$ to the conjugate point (that is, point having the same image) and in investigating the topological properties of the continuity set of f. *W. Hurewicz.*

Hall, D. W. and Puckett, W. T., Jr. Conditions for the continuity of arc-preserving transformations. Bull. Amer. Math. Soc. **47**, 468–475 (1941). [MF 4539]

If A is a locally connected continuum, a single-valued (not necessarily continuous) transformation T on A is said to be arc-preserving if $T(\alpha)$ is an arc or point for every arc α of A. If A is not cyclic it is simple to construct examples of arc-preserving transformations T, where $T(A)$ is not an arc and T is not a homeomorphism. This paper constructs an example for the cyclic case and proves the following theorem: Let T be arc-preserving, where A is cyclic and $T(A)$ is not an arc. Then T is a homeomorphism if (a) A is strongly arc-wise connected or (b) T is tree-preserving. A set A is strongly arc-wise connected if for every infinite subset K of A there is an arc α of A such that the set $\alpha \cdot K$ is infinite. *W. L. Ayres* (Lafayette, Ind.).

Wallace, A. D. Concerning relatively non-alternating transformations. Proc. Nat. Acad. Sci. U. S. A. **27**, 182–185 (1941).

The author considers a compact Hausdorff space M and a single valued continuous transformation $T(M)=N$. If $G=[Z]$ is a collection of closed subsets of N covering N, then T is said to be G-non-alternating provided that for no Z of G does $T^{-1}(Z)$ separate M between two points of $T^{-1}(y)$ for any point y of N. This definition reduces to the definition of non-alternating given by G. T. Whyburn [Amer. J. Math. **56**, 294–302 (1934)] in the case where G is the collection of all single points of N. The author states that T is monotone if and only if it is non-alternating with respect to the collection of all closed sets in N. In order that T be G-non-alternating each of the following conditions is both necessary and sufficient: If Z is a set of G and $M-T^{-1}(Z) = M_1+M_2$, where M_1 and M_2 are separated sets, then (i) $T(M_1)$ and $T(M_2)$ are separated sets, (ii) $M_i = T^{-1}T(M_i)$, for $i=1, 2$.

Using the idea of conjugate points [see Kuratowski and Whyburn, Fund. Math. **16**, 305 (1930)] in a form essentially the same as that employed by Radó and Reichelderfer [Duke Math. J. **6**, 474–485 (1940); these Rev. **1**, 318], the author defines a chain as any closed connected and nondegenerate subset C of the space S such that every point x of S conjugate to two distinct points of C is itself a point of C. A chain is a prime chain provided every pair of its points is conjugate. For any nondegenerate subset X of S the product of all chains in S containing X is a chain denoted by $C(X)$. Generalizing results previously obtained by G. T. Whyburn [loc. cit.] and G. E. Schweigert [see an abstract in Bull. Amer. Math. Soc. **44**, 636 (1938)], the author proves that, if M and N are compact connected Hausdorff spaces and $T(M)=N$ is non-alternating, then, for any set A in M, $C(T(A))$ is contained in $T(C(A))$, where the sets concerned are nondegenerate. The author states also that every non-alternating transformation $T(M)=N$, where M and N are connected, is non-alternating relative to the collection of all prime chains in N. If M and N are locally connected and metric and $T(M)=N$ is non-alternating, then (i) if N is a tree T is non-alternating relative to the collection of all subcontinua of N, (ii) if M is unicoherent the inverse of each A-set in N is a continuum. *D. W. Hall* (Providence, R. I.).

Chogoshvili, George. On Schnirelmann's transformations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 199–203 (1941). [MF 4266]

The author attributes to Schnirelmann "the problem of studying deformations of hypersurfaces of Riemann's n-dimensional manifolds, as run like the deformations of level surfaces of a non-degenerated function given on a manifold." The author has apparently not seen the paper "The analysis and analysis situs of regular n-spreads in $(n+r)$-space," Proc. Nat. Acad. Sci. U. S. A. **13**, 813–817 (1927), by Marston Morse. The paper begins by showing how the problem of deformations is equivalent from a topological point of view to the analysis of the changes in level surfaces of a function of $r+1$ variables when a critical point is passed. This analysis is a matter of local differential geometry. Once this equivalence is noted, the changes in the Betti numbers as given appear to be essentially the same as those listed by Morse in 1927. *M. Morse.*

Elsholz, L. Zu der Frage über die Bestimmung der unteren Grenze der Anzahl der kritischen Punkte einer stetigen Funktion, die auf einem Raum, der keine Mannigfaltigkeit ist, bestimmt ist. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 455–461 (1940). (Russian. German summary) [MF 3742]

[Note: The above title of the German summary is an incorrect translation of the Russian title, which reads: "On the problem of evaluation of the number of critical points of continuous functions defined over spaces which are not manifolds."] This paper is closely related to the article by the same author in Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 551–558 (1939) [cf. these Rev. **1**, 319]. In the present paper the author considers continuous functions defined over a locally compact continuum M in which every point with exception of a certain subset R has a neighborhood homeomorphic to the Euclidean E_n. The "singular set" R is assumed to consist of a finite number of components. The topological invariants which have been used to evaluate the number of critical points in the case of functions defined over a manifold (Lusternik-Schnirelmann category, homology category, Elsholz-Froloff length, etc.) could be used for the same purpose in the present case. They would give a very unprecise evaluation, however, and in order to obtain better results it is convenient to modify these invariants, relating them to the singular set R. For instance, it is convenient to replace the homotopy category cat A (where A is a subset of M) by the new invariant ${}_R\operatorname{cat} A$ defined as the least integer n with the property that A can be split into n subsets, each of which can be deformed over A into a single point in such a way that during this deformation the points of $A \cdot R$ describe trajectories entirely contained in R. *W. Hurewicz* (Chapel Hill, N. C.).

MECHANICS

Dynamics, Celestial Mechanics

Dimentberg, F. M. and Shor, J. B. Graphic solution of problems of spacial mechanics. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 5–6, 105–122 (1940). (Russian. English summary) [MF 4655]

Federhofer and Winter, employing the method of Mayor-Mises, give a graphic solution of many problems in spacial kinematics by a representation on one plane. The present work presents a solution of the problems of Federhofer and Winter, as well as of others, by a representation on one plane employing a procedure other than that of Mayor-Mises. This procedure is based on the authors' previous work. By employing this procedure, the representation of spacial systems becomes more visual, and the solution of problems simpler, even in more complicated cases.

Authors' summary.

de Mira Fernandes, A. A principle of minimal velocity. Portugaliae Math. **2**, 77–80 (1941). (Portuguese) [MF 4277]

The Hertz principle of least curvature in the configuration space is shown to lead to a principle of least velocity in the phase space. *D. C. Lewis* (Durham, N. H.).

de Mira Fernandes, A. Equazioni della dinamica. Portugaliae Math. **2**, 1–6 (1940). [MF 4275]

The equations of Appell and of Maggi for non-holonomic motions and of Lagrange for holonomic motions are derived from a principle essentially equivalent to the Gaussian principle of least constraint. The analogous facts for impulsive motion are also given. *D. C. Lewis.*

Pars, L. A. The action in a uniform field. Proc. Cambridge Philos. Soc. **37**, 168–176 (1941). [MF 4242]

The author considers the motion of a particle in the x, y-plane subject to a uniform force with potential $-gy$. The principle of least action asserts that the orbits joining a point $A:(x_0, y_0)$ to $B:(x_1, y_1)$ are the curves $x=x(t)$, $y=y(t)$ which give a stationary value to the integral $J=\int\{(y+k)(\dot{x}^2+\dot{y}^2)\}^{-\frac{1}{2}}dt$, where $k=h/g$ and h is the (constant) energy of the motion. If A is fixed and B is suitably situated, these extremals consist of (a) a pair of parabolas, and (b) the "Goldschmidt curve," composed of segments of the three lines $x=x_0$, $y+k=0$, $x=x_1$. The author shows that, depending on whether B lies inside or outside a certain curve Γ, the absolute minimum of J is given by one of the parabolas or by the Goldschmidt curve, respectively.

H. E. Robbins (New York, N. Y.).

Fleckenstein, J. O. Notiz sur Lagrangeschen Lösung des Keplerschen Problems. Comment. Math. Helv. **13**, 83–89 (1940). [MF 4438]

A proof is given for Bessel's theorem that, if the Kepler equation $t=x+\epsilon\sin x$ $(0\leqq\epsilon<1)$ is solved for x, then $x-t$ has a Fourier sine series expansion with coefficients of the form $2J_n(n\epsilon)/n$. The proof starts with the Lagrange formula and treats the derivative operators involved as algebraic quantities. *P. W. Ketchum* (Urbana, Ill.).

Sokoloff, George. Sur les pôles des coordonnées dans le mouvement symétrique d'un système de points matériels qui s'agissent avec des forces dépendantes des distances mutuelles. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1941**, no. 6, 33–50 (1941). (Ukrainian. Russian and French summaries) [MF 4498]

Points P_0, P_1, P_2 with masses m_0, m, m are considered, the force involved being $g^2 m_i m_j|f(\Delta_{i,j})|$, where $f(r)=F^{(1)}(r)$ is analytic for all $r>0$, while $f(r)r^{1-2\alpha}\rightarrow 2\alpha$ (as $r\rightarrow\infty$) and $f(0)$ is finite. The initial conditions are assumed to be such that $\overline{P_0P_1}=\overline{P_0P_2}$ $(=\rho)$ for all t. Trajectories are studied for which $\rho\rightarrow\infty$ (as $t\rightarrow t_1$; t_1 finite). The planar motion of P_1 relative to P_0 is considered. The axis of symmetry of the triangle $P_0P_1P_2$ is taken as the x-axis and the corresponding differential equations are set up. The author proves that, if the motion is regular for $t<t_1$ but not so for t_1, then $\rho\rightarrow\rho_1$ $(0\leqq\rho_1<\infty)$ and $y\rightarrow 0$, or $\rho\rightarrow\infty$ (as $t\rightarrow t_1$), in which case y or $1/y\rightarrow 0$. The investigation is given in somewhat greater detail on the basis of a suitably transformed system. The form of the solutions is given near $t=t_1$ when $r^{-2\alpha}F(r)$, $r^{1-2\alpha}f(r)$ are expressible as series in integral powers of $r^{-\alpha_1}, \cdots, r^{-\alpha_n}$ $(\alpha_2, \cdots, \alpha_n>0)$; in certain more general cases the asymptotic methods of Bohl and Cotton are shown to be adequate. *W. J. Trjitzinsky* (Urbana, Ill.).

Artemieff, N. Die Bestimmung der Realisierbarkeit der periodischen Bewegungen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 127–158 (1941). (Russian. German summary) [MF 4511]

A sufficient condition is given in order that trajectories for a class of periodic motions should be "realizable"; the latter concept has been introduced by the author previously [the same Bull. **1939**, 351–366 and 429–446; cf. these Rev. **1**, 281]. In a bounded domain G_n of the n-space of $(x)=(x_1, \cdots, x_n)$ he considers (1) $dx_i/dt=a_i(t, x)$ $(t\geqq 0)$, where the a_i, $\partial a_i/\partial x_k$ are continuous for (x), t in $G_{1,n}$ $\{0\leqq t$; (x) in $G_n\}$ and the $\partial a_i/\partial x_k\subset$ Lip. (in (x)); the corresponding perturbed system is taken in the form $dy_i/dt=a_i(y)+\epsilon b_i(t, y)$ $((y)=(y_1, \cdots, y_n)$; $|b_i|\leqq 1$; $t\geqq 0)$, where the $b_i\subset$ Lip. (in (y)). The main result is as follows. Let $(f(t))$ be a periodic solution of (1) of period 2π; let the c.e. (characteristic exponents) be $\lambda_k(\epsilon)$ (they are continuous in ϵ at $\epsilon=0$; the $\lambda_k(0)$ are the c.e. for $\epsilon=0$); suppose that the $\lambda_k(0)$ are purely imaginary and conjugate in pairs. When at least one of the $\lambda_k(\epsilon)>0$ for $0<\epsilon<\alpha$, then the periodic motion $(f(t))$ of (1) is positively non-realizable with respect to the perturbations b_i. Important applications are given to the study of conservative systems and to two problems of celestial mechanics. *W. J. Trjitzinsky* (Urbana, Ill.).

Speiser, Andreas. Topologische Fragen aus der Himmelsmechanik. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 204–213 (1940). [MF 4417]

In a simple case of the restricted problem of three bodies (Poincaré, G. D. Birkhoff), where the "asteroid" moves on a sphere with one attractive and one repulsive singularity and where the phase space is homeomorphic to projective space, it is shown that the singularities may be removed by suitable contact transformations. *H. E. Robbins.*

Lanczos, C. The dynamics of a particle in general relativity. Phys. Rev. (2) **59**, 813–819 (1941). [MF 4436]

The author defines the total momentum of a particle

described by the stress energy tensor T^{ik} (which vanishes outside and on the boundary of the particle) as $p^i=\{T^{i4}\}$ ($i=1, 2, 3$), where $\int X\sqrt{g}\, dx_1dx_2dx_3=\{X\}$. The total mass is defined as $M=\{T^{44}\}$. The divergence relation satisfied by the stress energy tensor, namely,

$$\frac{\partial\sqrt{g}T^{\beta\alpha}}{\partial x^\alpha}=-\Gamma^\beta_{\mu\nu}T^{\mu\nu}$$

is then shown to give

$$(1)\qquad \frac{dp^i}{dx^4}=F^i=-\{T^{\mu\nu}\Gamma^i_{\mu\nu}\},\quad \frac{dM}{dx^4}=F^4=-\{T^{\mu\nu}\Gamma^4_{\mu\nu}\}.$$

The F^α are then interpreted as the "moving force." It is next shown that as a consequence of the divergence relations

$$(2)\qquad p^i=M\frac{d\zeta^i}{dx^4}+T^{\mu\nu}\Gamma^4_{\mu\nu}(x_i-\zeta^i)=M\frac{d\zeta^i}{dx_4}+\bar{p}^i,$$

where $\zeta^i=\{x_iT^{44}\}/M$, and are interpreted as the coordinates of the center of gravity of the particle. It is claimed that the term $\bar{p}^i$ "can only be a small correction term which can contribute only negligible amounts to the law of motion" and this term is neglected in further considerations.

The "moving force" is shown to be a surface integral except in the static case. In the non-static case it is stated that the part which is not a surface integral cannot contribute to the stationary part of the moving force. The author then shows that if a static particle is acted on by an external field which does not change T^{ik} for the particle, and such that the derivatives of the external field are constant over the particle, and if $\bar{p}^i$ is neglected, then the law of motion given by (1) and (2) is such that in the second approximation the center of gravity travels along a geodesic. In case the stress energy tensor T^{ik} satisfies the further condition $T_i{}^i=T=0$, then the author states, on the basis of a previous work [Phys. Rev. (2) 59, 708–716 (1941)], that the center of gravity does not follow a geodesic, but that its acceleration is less. The author considers that this may be a clue to the understanding of the anomalous value of the light deflection on the limb of the sun, discovered by Freundlich and his collaborators. However, his theoretical prediction is more than twice as large as that indicated by the experiments. It is not evident that the results of the previous paper apply to the paper under review.

A. H. Taub (Seattle, Wash.).

Eakin, W. C. H. and McCrea, W. H. Velocity distributions in a field of force. Proc. Roy. Irish Acad. Sect. A. 46, 91–102 (1940). [MF 3054]

A distribution of moving particles whose collisions are negligible and whose gravitational field is zero is acted on by a central field of force centered at some point within the distribution. The distribution is of infinite extent and the field of force is zero at infinity. It is first proved that if the velocity distribution of the particles is isotropic at infinity it is isotropic everywhere. In the case when the field of force is due to a gravitating sphere, a formula for the particle density at distance r from the center of this sphere is worked out. The number of particles striking the sphere is also given, both for a sphere at rest relative to the mean standard of rest of the particles at infinite distance, and for a sphere with a given velocity relative to this standard. The relationship of the investigation to that of Mott-Smith and Langmuir on the theory of exploring electrodes in discharge tubes is also worked out.

G. C. McVittie (London).

Heckmann, O. Zur Kosmologie. Nachr. Ges. Wiss. Göttingen. Fachgruppe II. (N.F.) **3**, 169–181 (1940). [MF 4449]

This paper deals with dynamic cosmology, to be distinguished both from relativistic and Milne's kinematic cosmologies. The concepts of classical dynamics and Newton's gravitational law are assumed; the density, velocities, pressure, potential are assumed to be bound by the hydromechanical equations of motion. The added world-postulate is (roughly speaking) that all observers have the same world-view. The world-models which can be built from these hypotheses are essentially the same as those in relativistic cosmology and exactly the same when the pressure vanishes. The treatment of optical phenomena in the dynamic cosmology encounters difficulties which lead the author to discuss seriously the hypothesis of an ether in which all observers are at rest. The last section of this paper is devoted to a polemic against the theory of Gamow and Teller [On the origin of great nebulae, Phys. Rev. (2) **55**, 654–657 (1939)].

L. Infeld (Toronto, Ont.).

Hydrodynamics, Aerodynamics

Wieghardt, K. Zusammenfassender Bericht über Arbeiten zur statistischen Turbulenztheorie. Luftfahrtforschung **18**, 1–7 (1941). [MF 4612]

Kolmogoroff, A. The local structure of turbulence in incompressible viscous fluid for very large Reynolds' numbers. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941). [MF 4385]

The author defines local homogeneity and local isotropy for the components $w_\alpha(P)=u_\alpha(P)-u_\alpha(P^{(0)})$, $\alpha=1, 2, 3$, where the $u_\alpha(P)$ are the velocity components at the point $P=(x_1, x_2, x_3, t)$ and $P^{(0)}$ is a fixed point. The $u_\alpha(P)$ are assumed to be random variables. These definitions are stated in terms of the distribution law F_n of the $w_\alpha(P)$. The definition of local isotropy is narrower than that of isotropic turbulence in the sense of G. I. Taylor [Proc. Roy. Soc. London. Ser. A. **151**, 421–478 (1935)] in that steadiness in time is required; it is broader since it applies to velocity differences rather than to velocities themselves. The author believes that his definitions may be realized with good approximation in many cases of practical interest. He proceeds to consideration of the correlation tensor and dissipation of energy and obtains some formulae analogous to those for isotropic turbulence. Finally the behavior of the correlation function is investigated on the basis of two successive hypotheses of similarity.

W. R. Sears.

Ursell, H. D. The motion of a solid through an infinite liquid under no forces. Proc. Cambridge Philos. Soc. **37**, 150–167 (1941). [MF 4241]

The paper is based on Kirchhoff's equations of motion for a solid in a perfect fluid, and deals with questions of steady motion and stability. All characteristics of motions depend on the 21 coefficients in the quadratic form expressing the kinetic energy T in terms of the 6 components of linear and angular velocity of the solid. These coefficients must be such that T is positive-definite; the author believes that they are also subject to other non-obvious inequalities, but leaves this question open. Two methods are used: (i) direct deductions from the positive-definite character of

T, (ii) construction of approximate forms for T by attaching small circular discs rigidly to a sphere, at a great distance from it. By (i) the author reestablishes by a more thorough argument the known result that one of the three steady motions of translation is stable. Further, he investigates the existence and stability of steady screw motions, finding in particular that a stable translation may be rendered unstable by superposing a rotation. By (ii) he shows that (contrary to a statement commonly made) two of the steady translations of a solid may be stable. The approximations in method (ii) are not completely explained, no mention being made of the fact that the velocity is infinite at the edge of a moving disc. *J. L. Synge* (Toronto, Ont.).

Bechert, Karl. Zur Theorie ebener Störungen in reibungsfreien Gasen. II. Ann. Physik (5) **38**, 1–25 (1940). [MF 4420]

[Part I appeared in the same journal (5) **37**, 89–123 (1940); cf. these Rev. **1**, 286.] The equation of Darboux

$$V_{\xi\xi}+(2m/\xi)V_{\xi}=V_{\eta\eta}$$

is solved for an arbitrary value of m with the aid of operators L, M and generalized derivatives. If $v=\xi+\eta$, $w=\xi-\eta$, the formula for a non-integral value of m is

$$V=\sum_{n=0}^{\infty}[(-)^n(m-n)\Gamma(m+n)2^{-n}/2\pi i(n!)]$$
$$\times\left[\int\Phi(z)dz(z-v)^{n-m-1}+\int\Psi(z)dz(z-w)^{n-m-1}\right],$$

where the integration is in each case on the Riemann surface of the integrand and round a closed curve surrounding the point v or w. Much of the analysis deals with the generalization of repeated operators of type $L=(1/\xi)D_{\xi}$, $M=D_{\xi}\cdot\xi^{-1}$, where D_{ξ} denotes a differentiation with respect to ξ. An extension is given also of the relation $D_{\xi}L^N=M^ND_{\xi}$, which is valid for all integral values of N. The analysis is used to solve the problem of standing waves between the planes $x=\pm a$, and it is found that for the case of finite amplitude the nodes, where the velocity is zero, have the same location as when the amplitude of vibration is very small compared with a. Just as in the case of steady motions the solution of the wave problem is particularly simple when $m=-1$, that is, when the relation between p and ρ is of type $p=p_0-c^2/\rho$. *H. Bateman* (Pasadena, Calif.).

Sakadi, Zyuro. On the extension of the differential equations of incompressible viscous fluid. Proc. Phys.-Math. Soc. Japan (3) **23**, 27–33 (1941). [MF 3909]

The author assumes generalized nonlinear stress rate of strain relations, on the basis of which he solves some flow problems. In a "correction added in proof" he states that his generalization is of a physically impossible nature. *E. Reissner* (Cambridge, Mass.).

Davies, T. V. An investigation of the flow of a viscous fluid past a flat plate, using elliptic coordinates. Philos. Mag. (7) **31**, 283–313 (1941). [MF 4478]

An infinitely long strip of flat plate with width $2c$ and vanishingly small thickness is placed in a stream of viscous fluid with velocity V parallel to the plate at infinity. This two-dimensional problem is solved by using Oseen's approximation, in which the difference between viscous flow and non-viscous flow is assumed to be small and second order quantities are neglected. The method of attack is that of T. Lewis [Quart. J. Math., Oxford Ser. **9**, 21–31 (1938)], using elliptic coordinates. The drag on the plate is shown to be associated with the multivalued solutions of the equation

$$\left(\frac{\partial}{\partial\alpha}+\nu\nabla^2\right)U=0.$$

This is proved to be in accordance with the Filon-Goldstein theorem, which expresses the drag in terms of the inflow at the wake. The stream function is then expressed as a series involving Mathieu and associated Mathieu functions. By Fourier expansion of the Mathieu function at the surface of the plate, the boundary conditions are expressed in a set of infinite number of linear algebraic equations of the undetermined coefficients in the expansion of stream function. The drag is calculated as the ratio of two infinite determinants whose elements are functions of the Reynolds number $k=Vc/2\nu$, ν being the kinematical viscosity. It is stated that there are solutions of the fundamental equation which are not included in the stream function used and which can be made to satisfy the boundary conditions independently. These solutions will not affect the drag, but will change the flow. This indicates that there is necessarily a great degree of arbitrariness in the solutions of all problems involving the flow of a viscous fluid past solids under Oseen's approximation. For small values of k, the drag is obtained explicitly as a function of k up to second approximation. The first approximation agrees with that of Piercy and Winny [Proc. Roy. Soc. London. Ser. A. **140**, 543–561 (1933)] who also used Oseen's approximation but a different method of attack. The second approximations, however, differ slightly. Piercy and Winny obtained an asymptotic solution for large values of k which is 70 per cent in excess of the more exact solution of Blasius. In the appendices, some results of investigation in associated Mathieu functions are given. *H. S. Tsien* (Pasadena, Calif.).

Ginsburg, I. P. On the question of motion of real gases at large velocities. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **7**, 5–60 (1939). (Russian) [MF 3335]

The author studies the equations of motion of a real gas with the equation of state $(p+\alpha/V^2)(V-C)=RT$. The following questions are considered in the paper: (1) Using the general equations of mechanics and thermodynamics, the author establishes the equations of motion and gives the formulation of the theorem of Bernoulli. (2) He considers the one dimensional stationary motion and determines how the critical speed depends upon the coefficients α and C. (3) Using the general results on propagation of disturbances in a homogeneous medium, he determines the speed of sound. Further, the equations of the surfaces of discontinuity of velocity, of pressure and of density in the case of a weak and of a strong discontinuity are established. (4) He finds the general methods for the solution of plane and of one dimensional non-stationary problems for a real gas. (5) Using the theory of characteristics he considers the problem of propagation of waves caused by explosion. *S. Bergmann* (Cambridge, Mass.).

Küchemann, Dietrich und Vandrey, Friedrich. Über den Einfluss der Düse (oder des Auffangtrichters) auf Widerstandsmessungen im Freistrahl. Z. Angew. Math. Mech. **21**, 17–31 (1941).

Given an infinite plane to which is attached a nozzle of circular cross section. The authors investigate the flow from this nozzle for velocities which are derivable from a poten-

tial. After making some remarks upon the potential theory aspects of the problems, velocity components are computed and boundary conditions are introduced. This leads to an integral equation of the second type which may be solved numerically by iteration. Finally some remarks are made about the physical picture at hand. *A. E. Heins.*

Söhngen, Heinz. Bestimmung der Auftriebsverteilung für beliebige instationäre Bewegungen (Ebenes Problem). Luftfahrtforschung 17, 401–420 (1940). [MF 4051]

The distribution of forces on a thin airfoil in non-uniform two-dimensional motion is determined. The motion is arbitrary, provided that the deviation of the airfoil and its trail of vortices from a plane can be neglected, and includes cases in which the airfoil passes through arbitrary vertical-gust profiles. The system of equations to be solved in any case consists of the following: (1) The boundary condition at the airfoil:

$$(1)\quad \int_{-1-s}^{-1}\epsilon(\xi,s)(x-\xi)^{-1}d\xi+\oint_{1}^{1}\Gamma(\xi,s)(x-\xi)^{-1}d\xi+2\pi w(x,s)=0,$$

where $s(t)$ is the distance from a fixed point to the trailing edge ($x=-1$) at any time t, $\epsilon(x,s)$ and $\Gamma(x,s)$ are the vortex strengths in the wake ($x<-1$) and on the airfoil ($-1<x<1$), respectively, and $w(x,s)$ is the normal component of the velocity of the fluid relative to the airfoil. (2) The condition of vanishing total circulation of the system:

$$(2)\quad \int_{-1-s}^{-1}\epsilon(\xi,s)d\xi+\int_{-1}^{1}\Gamma(\xi,s)d\xi=0.$$

(3) The condition of finite velocity at the trailing edge:

$$(3)\quad \epsilon(-1,s)=\Gamma(-1,s).$$

(4) The assumption that the free vortices are stationary in the fluid after they leave the trailing edge:

$$(4)\quad \epsilon(x,s)=\epsilon(-1,s+x+1).$$

Equations (1) to (4) are to be solved for given $w(x,s)$ and $s(t)$. From the solution $\Gamma(x,s)$ the force distribution is calculated; the force is proportional to $\gamma(x,s)$, which is defined by

$$(5)\quad \gamma(x,s)=\Gamma(x,s)-\frac{\partial}{\partial s}\int_{1}^{x}\Gamma(\xi,s)d\xi.$$

The author uses an inversion formula of an earlier paper for the integral equation

$$\oint_{-1}^{1}f(\xi)(x-\xi)^{-1}d\xi=2\pi g(x),\qquad -1<x<1,$$

to obtain an integral expression for $\Gamma(x,s)$ involving $w(x,s)$ and a function $a(s)$ dependent upon $w(x,s)$ and $\epsilon(x,s)$. In any specific problem $a(s)$ is evaluated by an integration using the "Wagner function" $k_1(s)$ which gives the force on an airfoil that is stationary for $t<0$ and moves with constant velocity and altitude for $t>0$. Since $k_1(s)$ cannot be exactly expressed in usable form, approximate expressions given by von Kármán and Sears [J. Aeronaut. Sci. 5, 379–390 (1938)] and by I. Garrick [Nat. Adv. Comm. Aeronaut., Tech. Rep. 629 (1938)] are suggested.

It is shown that a sufficient condition for solubility of equations (1)–(4) by a unique pair of functions $\Gamma(x,s)$, $\epsilon(x,s)$ is that the function

$$\Gamma_0(s)=2\int_{-1}^{1}w(x,s)((1-x)/(1+x))^{\frac{1}{2}}dx$$

be stepwise continuous and differentiable. This function represents the circulation that would be calculated for the airfoil by the equations of the usual steady-motion thin-airfoil theory; that is, it is the "quasi-steady circulation." Expressions are given for the force and couple in a general case; these are equivalent to those of von Kármán and Sears [ibid.]. In the case of harmonic motion the force distribution is the same as calculated by Küssner [Luftfahrtforschung 13, 410–424 (1936)]. The more general results of the present paper are applied to the cases of a wing with a flap, with and without a gap between the flap and the main wing. As an example the load distributions are plotted for a wing with a 50%-chord plain flap deflected 200 degrees per second at various flight speeds. A comparison is also made between the rates of increase of lift due to flap deflection at constant angular velocities as calculated by the author's formulae and as measured experimentally in Göttingen. The agreement between theory and experiment is remarkably good. The author's results are also applied to an investigation of the common concept of the "critical speed" in flutter calculations. This is defined as the flight speed at which an undamped harmonic oscillation of an elastically suspended wing with flap ultimately results after a disturbance; it is supposed to separate regimes of positive and negative damping. The integrodifferential equations of such a system are set up and their behavior for large s, after a disturbance at $s=0$, is investigated by the methods of the Laplace transform. It is found that the existence of a critical speed depends solely on the existence of zeros of a transcendental function $f(z)$, involving $k_1(s)$, in the half-plane $\Re(z)>0$. A subsident motion cannot result after a disturbance at such a critical speed. However, it is possible for a critical speed to occur within a regime of positive or negative damping. It is proposed that the von Kármán-Sears approximation to $k_1(s)$ be used in this connection; the function $f(z)$ then becomes a polynomial.

Appendices to the rather exhaustive paper include a brief resume of Laplace-transform theory, a calculation of $k_1(s)$ by the use of this theory and the determination of several properties of the "Theodorsen function" $K_0(z)\{K_0(z)+K_1(z)\}^{-1}$, where $K_0(z)$ and $K_1(z)$ are modified Bessel functions of the second kind. *W. R. Sears.*

Giovannozzi, Renato. Un metodo generale per la determinazione del fattore di interferenza nelle gallerie aerodinamiche a contorno libero, rigido e misto. Pont. Acad. Sci. Comment. 3, 545–592 (1939). [MF 4104]

The determination of the interference of the wall of a wind-tunnel on an aerofoil is treated as a plane problem in the plane of the cross-section. The trailing vortices are replaced by a doublet in this plane, and the wall is represented by a closed curve, consisting of free and rigid portions. The problem is to find a complex potential $\Phi(z)=\phi(x,y)+i\psi(x,y)$ with the required singularity at the doublet and satisfying on the wall the conditions (1) $\phi=$constant on the free portions, (2) $\psi=$constant on the rigid portions. The plan of the paper is to transform the section conformally into the upper half of a Z-plane, the real axis corresponding to the wall. Then, for a general analytic function

$F(Z)=f(X, Y)+ig(X, Y)$, we have

$$(*)\qquad F(Z)=(1/\pi i)\int_{-\infty}^{\infty}(f(x, 0)/(x-z))dX;$$

this expresses $F(Z)$ at any point in terms of the values of its real part on the X-axis. Except in very simple cases, this formula does not give the required potential directly, and an auxiliary function $W(Z)$ is introduced. If $X_1, X_2, \cdots, X_{2p+2}$ are the points of junction on the X-axis of the free and rigid portions, the function

$$W(Z)=\frac{a_0Z^p+a_1Z^{p-1}+\cdots+a_p}{((Z-X_1)(Z-X_2)\cdots(Z-X_{2p+2}))^{\frac{1}{2}}}$$

(where $a_0, \cdots, a_p$ are real constants) is alternately real and pure imaginary as we pass through the free and rigid portions of the X-axis. Hence, if the constant values of ϕ and ψ on the free and rigid portions are supposed known, the real part of $\Phi(Z)W(Z)$ is known right along the X-axis, and so (*) may be used to obtain $\Phi(Z)W(Z)$, and hence $\Phi(Z)$. The method is applied to special cases where the wall is circular and the free and rigid portions symmetric with respect to the axis of the doublet. Some results of Kondo, obtained by a different method, are verified [cf. Rep. Aeronaut. Research Inst. Tôkyô Imp. Univ. 10, 197–259 (1935)].

J. L. Synge (Toronto, Ont.).

Pistolesi, E. Sull'interferenza di una galleria aerodinamica a contorno misto. Pont. Acad. Sci. Comment. **4**, 321–341 (1940). [MF 4112]

For the case of a circular wall, the author gives a more direct method of determining the interference than that of Giovannozzi [see preceding review]. He uses the complex velocity $w(z)$ instead of the complex potential; the boundary conditions are that the velocity shall be normal on the free portions and tangential on the rigid portions. (These are equivalent to the conditions of Giovannozzi.) Hence it follows that $zw(z)$ is real on the free portions and pure imaginary on the rigid portions. This fact enables the author to write down solutions for the simpler distributions of free and rigid portions, and leads him to a general formula. As a sample solution, the following may be quoted: with a doublet at the origin, pointing along the y-axis, and the wall a unit circle with center at the origin, rigid for $-b<\theta<b$ and the rest free, the complex velocity is

$$w(z)=(iA/z)(\cos\tfrac{3}{2}\zeta-\cos b\cos\tfrac{1}{2}\zeta)(\cos\zeta-\cos b)^{-\frac{1}{2}},$$

where $\cos m\zeta=\frac{1}{2}(z^m+z^{-m})$, and A is a real constant.

J. L. Synge (Toronto, Ont.).

Ziller, Felix. Beitrag zur Theorie des Tragflügels von endlicher Spannweite. Ing.-Arch. **11**, 239–259 (1940). [MF 4218]

The problem of determining the distribution of the lift over the span of a wing of given geometrical configuration according to the Prandtl lifting-line approximations is formulated as a variation problem. An integral expression is set up whose minimal condition leads to an integral equation of the second kind with symmetrical kernel; this can be identified with Prandtl's integral equation. The variation problem is attacked by the Ritz method, assuming the lift-distribution function to be expressible in a sine series. The infinite system of linear equations then obtained for the Fourier coefficients thereof is the same as is solved approximately in the well-known Lotz procedure. It is treated in some detail in the special cases of elliptical, rectangular and trapezoidal planforms and a family of planforms lying between the rectangular and the elliptical.

In the second part of the paper the problem of the lift distribution on a torsionally elastic wing is attacked by an analogous method. The variation problem is formulated and the Ritz method leads to another infinite set of linear equations for the Fourier constants of the lift distribution. Its form is exactly the same as for the rigid wing, but the coefficients depend on the elastic properties of the wing and the zero-lift moment coefficients of the profiles as well as on the planform and twist. The author discusses the determination of the elastic properties for single-spar wings with torsion tubes and two-spar wings. Finally, as an example, the case of a rectangular elastic wing is treated, assuming the single-spar type of design and a simple variation of torsional stiffness along the span. The coefficients appearing in the linear-equation system for this case are calculated in the form of corrections to be added to the corresponding coefficients for the rigid wing, which were calculated earlier.

W. R. Sears (Inglewood, Calif.).

Küssner, H. G. Allgemeine Tragflächentheorie. Luftfahrtforschung **17**, 370–378 (1940). [MF 4050]

On the assumption of small disturbances, the velocity potential in a frictionless compressible fluid satisfies the wave equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\phi}{c^2\partial t^2}=0.$$

For example, the potential for a stationary source at the origin is $\phi_0=(1/r)f(t-r/c)$, where $r=(x^2+y^2+z^2)^{\frac{1}{2}}$ and c is the velocity of sound. The author applies to this expression the Lorentz transformation $x=(X'+vt')/(1-v^2/c^2)^{\frac{1}{2}}$, $y=y'$, $z=z'$, $t=(t'+vX'/c^2)/(1-v^2/c^2)^{\frac{1}{2}}$; this yields a function with a singularity at $X'=-vt'$, which satisfies the equation

$$\frac{\partial^2\phi}{\partial X'^2}+\frac{\partial^2\phi}{\partial y'^2}+\frac{\partial^2\phi}{\partial z'^2}-\frac{\partial^2\phi}{c^2\partial t'^2}=0,$$

since the wave equation is invariant in the transformation. Hence this is the potential for a source moving with velocity v in the negative X' direction relative to the stationary X', y', z' coordinate system, when the undisturbed fluid is at rest relative to this system. He then transforms to a coordinate system moving with the singularity: $x'=X'+vt'$, y', z', t'. Anticipating that a lifting wing will be replaced by a dipol distribution, he obtains a formula for the pressure field of a moving dipol:

$$p(x', y', z', t')=-\rho\frac{\partial}{\partial t'}\frac{\partial\phi_0}{\partial n}=-\rho\frac{\partial}{\partial n}\left\{\frac{1}{r}\gamma\left(t-\frac{r}{c}\right)\right\},$$

where n is the normal to the wing surface, ρ is the fluid density and γ is an arbitrary function. An independent relation for ϕ is then obtained in terms of the pressure integrated from $x'=-\infty$ to x' in the manner of Prandtl's "acceleration potential" theory [Z. Angew. Math. Mech. **16**, 360–361 (1936)]. By combining these expressions the author arrives at an integral equation independent of ρ for the function γ, which is proportional to the pressure discontinuity at the surface. The known functions characterize the geometry of the lifting surface and its motion relative to the undisturbed fluid.

This integral equation has a rather complicated form in general. The author simplifies it for several special cases:

(1) For a wing of infinite span in two-dimensional harmonic motion it reduces to "Possio's equation," an integral equation not explicitly given by Possio [Aerotecnica **18**, 441–458 (1938)] but deducible from his results. In the sub-case of incompressible flow ($c=\infty$), this reduces to a form used by Birnbaum. In the alternate sub-case of steady motion, it reduces to a well-known expression of the two-dimensional thin-airfoil theory modified according to the Prandtl-Glauert approximation for a compressible fluid. (2) For a wing of infinite span with conditions varying sinusoidally along its span, in an incompressible fluid, the equation reduces to a simple form. (3) For wings of finite span in harmonic motion in an incompressible fluid the integral equation does not appear tractable in general. For wings of large aspect ratios and similar cases, the author makes certain approximations and obtains a simpler form, which reduces to the well-known Prandtl lifting-line integral equation when the chord length is made to vanish and the period of oscillation is infinite. However, this simpler form is valid only when the relative normal velocity at any point is proportional to exp $(t-x/v)$, as when a rigid wing passes through a stationary gust pattern. If the wing is oscillating or deforming harmonically while moving through a fluid at rest, a choice must be made of a "typical" relative velocity for each chordwise section at each instant. The author proposes an approximate method of accomplishing this. An alternative procedure of successive approximations is suggested for solution of the original integral equation if more accuracy is desired. (4) The author closes with a brief discussion of the application of the general integral equation to biplanes, wings with slots and other multiplane arrangements. A method of successive approximations is proposed.

The paper includes three tables of numerical values of functions involved in the solution of various problems. These are

$$T(\omega)=\frac{H_0^{(2)}(\omega)+iH_1^{(2)}(\omega)}{-H_0^{(2)}(\omega)+iH_1^{(2)}(\omega)}, \qquad \text{for } 0\leqq\omega\leqq\infty,$$

$$U_1(s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{T(-i\omega)}{\omega}\exp(\omega s)d\omega, \qquad \text{for } 0\leqq s\leqq\infty,$$

and

$$S(x)=(i\pi/2)x\int_x^\infty G_1(u)du/u, \qquad \text{for } 0\leqq x\leqq\infty,$$

where $H_0^{(2)}(\omega)$ and $H_1^{(2)}(\omega)$ are Hankel functions, and $G_1(u)=Z_1(iu)-N_1(iu)-2/\pi$, $Z_1(x)$ and $N_1(x)$ are Struve and Neumann functions, respectively. *W. R. Sears.*

Küssner, H. G. General airfoil theory. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **979**, 27 pp. (1941). (1 plate) [MF 5041]

English translation of the paper reviewed above.

Krienes, Klaus. The elliptic wing based on the potential theory. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 971, 43 pp. (1941). (3 plates) [MF 5042]

English translation of the author's paper published in Z. Angew. Math. Mech. **20**, 65–88 (1940); these Rev. **2**, 27.

Binnie, A. M. Waves in an open oscillating tank. Engineering **151**, 224–226 (1941). [MF 4255]

The author develops an approximate theory for the waves formed in an inviscid liquid contained in a rectangular tank subjected to small angular oscillations about a fixed axis O which is perpendicular to a pair of sides of the tank, the angular displacement being defined by $\theta=\theta_0 \sin pt$. The treatment is two-dimensional. Choosing O as origin and axes of coordinates (x, y) which rotate with the tank, the author supposes that the velocity potential ϕ has the form $\phi=\phi_1\cos pt$, where ϕ_1 is a function of x and y only. The condition imposed on ϕ at the surface of the fluid is that normally used with fixed axes, but the error resulting from the oscillation is small for sufficiently small θ_0.

D. C. Spencer (Cambridge, Mass.).

Srettensky, L. N. On the gravitational oscillations of gaseous spheres. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 5–6, 87–104 (1940). (Russian. English summary) [MF 4654]

The work deals with the question of small oscillations of a spherical mass of gas within the field of its own gravitation. By employing the methods of the theory of sound and the wave theory, equations of oscillations of the spherical mass of gas relative to its hydrostatic state of equilibrium are established. A study of the frequency of oscillations leads the author to the conclusion that a gaseous sphere of any radius will be stable, provided changes in the field of Newtonian potential arising from the redistribution of mass of the oscillating gas are not taken into consideration. If the aforesaid changes of Newtonian potential are taken into account, only those spheres will be stable whose radii satisfy an additional condition. *Author's summary.*

Frössling, Nils. Verdunstung, Wärmeübergang und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminarer Grenzschichtströmung. Lunds Univ. Årsskrift (N.F.) **36**, no. 4, 32 pp. (1940) = Fysiogr. Sällskapets Handlingar (N.F.) **51**, no. 4, 32 pp. (1940). [MF 4525]

The equations for laminar boundary-layer flow and for evaporation (transport of a substance) and heat transfer therein are solved for two-dimensional and axially symmetric cases. The method used is that of Blasius [Göttingen dissertation, 1907], Hiemenz [Göttingen dissertation, 1911] and Howarth [Aeron. Res. Com. Reports and Memoranda, no. 1632 (1935)], in which the velocity profile is developed in powers of x, the distance along the surface from the forward stagnation point. The same treatment is here applied to the concentration and temperature profiles. Following Howarth, the author obtains the results in the form of functions independent of the Reynolds number and of the geometry of the surface, so that they are applicable to all cases. Certain of these functions given by Howarth for the two-dimensional boundary layer have been calculated more accurately by the author. He also tabulates the analogous functions for the axially symmetric boundary layer and for the two-dimensional and axially symmetric transport layers, corresponding to the first three terms in a power series in x. Methods of determining the error due to termination of the series are discussed briefly. A step-by-step method of continuing the two-dimensional boundary layer downstream from any point has been given by Prandtl [Z. Angew. Math. Mech. **18**, 77 (1938)]. The author reviews this method and gives analogous methods for the other cases under consideration. It is shown that for large Reynolds numbers the Nusselt number is proportional to the cube root of Prandtl's number. Finally, certain approximate methods used in boundary-layer calculations, which involve assumptions as to the type of velocity profile present, are discussed. *W. R. Sears* (Inglewood, Calif.).

Theory of Elasticity

Papkovitsch, P. F. Über eine Form der Lösung des byharmonischen Problems für das Rechteck. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 334–338 (1940). [MF 3249]

This is a contribution to the classical problem of finding the solution of the bi-harmonic equation $\nabla^2\nabla^2\varphi=0$ with prescribed values of φ and $\partial\varphi/\partial n$ along the boundary of a plane rectangular region. In view of the fact that it is simple to reduce the problem to the case where φ and $\partial\varphi/\partial n$ vanish along one pair of opposite boundary lines, the author considers the problem for the semi-infinite strip $|y|\leqq b$, $0\leqq x$, with prescribed boundary value along the side $x=0$ and zero boundary values along the sides $y=\pm b$. Assuming, in generalization of Maurice Lévy's well-known solution,

$$\varphi=\sum\{a_kY_k(y)e^{-\lambda_kx}+\bar{a}_k\bar{Y}_k(y)e^{-\bar{\lambda}_kx}\},$$

the constants of integration contained in the Y_k and the complex λ_k are determined from the conditions along $y=\pm b$. The remaining complex constants a_k are determined from the two conditions along $x=0$, utilizing certain orthogonality relations between the Y_k and their second derivatives. The solution for the finite rectangle is obtained from this solution by a rapidly converging process of iteration.

The author's method is essentially the same (except for the way the constants a_k are determined) as the one published independently by Fadle [Ing.-Arch. **11**, 125–149 (1940); these Rev. **2**, 30]. *E. Reissner.*

Steuermann, E. A generalization of Hertz' theory of local deformations in elastic bodies pressed against each other. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 179–181 (1940). [MF 3886]

The author derives a vector integral equation for the solution of the problem of two elastic bodies in contact. It is shown that the classical theory of Hertz is obtained by making appropriate simplifying assumptions (essentially this consists in replacing the surfaces in contact with their osculating paraboloids). *J. J. Stoker* (New York, N. Y.).

Steuermann, E. Local deformations in elastic circular cylinders with nearly equal radii under pressure. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 182–184 (1940). [MF 3887]

The method of the author [see the preceding review] is applied to the special problem indicated in the title and the results obtained are compared with those found by the method of Hertz. For the most part the results by the two methods are the same. The method of Hertz, however, yields values much too large for the total pressure between the two surfaces when the area of contact is large.
J. J. Stoker (New York, N. Y.).

Maisel, W. M. A generalization of the Betti-Maxwell theorem to the case of thermal stressed condition and some cases of its application. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 115–118 (1941). [MF 4264]

Let $\mathbf{u}_1$ be the displacement vector in a homogeneous isotropic elastic solid due to body forces $\mathbf{Q}_1$, surface forces $\mathbf{P}_1$ and a temperature distribution T_1. Let $\mathbf{u}_2$ be the displacement vector due to $\mathbf{Q}_2$, $\mathbf{P}_2$ and T_2. The author's result is the formula

$$\iiint\mathbf{Q}_1\cdot\mathbf{u}_2dV+\iint\mathbf{P}_1\cdot\mathbf{u}_2dS+\frac{\alpha E}{1-2\nu}\iiint T_1\operatorname{div}\mathbf{u}_2dV=\iiint\mathbf{Q}_2\cdot\mathbf{u}_1dV+\iint\mathbf{P}_2\cdot\mathbf{u}_1dS+\frac{\alpha E}{1-2\nu}\iiint T_2\operatorname{div}\mathbf{u}_1dV.$$

For $T_1=T_2=0$ this reduces to Betti's reciprocity theorem.
E. Reissner (Cambridge, Mass.).

Beguiachvili, A. I. Solution du problème de la pression d'un système de profiles rigides sur la frontière rectiligne d'un demiplan élastique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 914–916 (1940). [MF 3587]

Muschelišvili gave a method for the solution of the problem of a rigid body supported by the elastic half plane along one single interval of the real axis [Some fundamental problems of the mathematical theory of elasticity (Russian), 1935]. [See also M. Sadowsky, Z. Angew. Math. Mech. **10** (1930).] The author generalizes this method to the case when there are several bodies rigidly connected and the line of contact consists of several intervals. The problem is reduced to the integral equation $\int_S p(t)\lg|t-x|dt=Q(x)$, where Q is a known function. The author gives an explicit solution of the equation above. The case when the line of contact consists of two pieces is considered more in detail.
S. Bergmann (Cambridge, Mass.).

Michlin, S. Sur un problème particulier de la théorie de l'élasticité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 535–536 (1940). [MF 3224]

In terms of functions of a complex variable there is given a solution for the plane stresses in an infinitely extended orthotropic sheet having a finite number of slits along one straight line and a self-equilibrating system of forces on the border of each slit. *E. Reissner* (Cambridge, Mass.).

Sakadi, Zyuro. Plastic deformation of a circular cylinder and spherical wave in plastic solid. Proc. Phys.-Math. Soc. Japan (3) **23**, 33–37 (1941). [MF 3910]

Assuming as relations between stresses σ and strains ϵ

$$\frac{1}{\tau}\sigma_{ij}+\frac{\partial\sigma_{ij}}{\partial t}=\lambda\delta_{ij}\frac{\partial}{\partial t}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33})+2\mu\frac{\partial\epsilon_{ij}}{\partial t}+\frac{1}{3\tau}(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{ij},$$

where $\delta_{ij}=0$ when $i\neq j$ and $\delta_{ij}=1$ otherwise, the author treats the problem of the torsion of a circular cylinder and the problem of the propagation of a spherical dilatational wave. *E. Reissner* (Cambridge, Mass.).

Konovalov, J. V. Flexure of a thin-wall cylindrical shell. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) 4, no. 5–6, 35–54 (1940). (Russian. English summary) [MF 4651]

The work deals with the approximate solution of the problem of flexure and stability of a circular cylinder of infinite length with an invariable contour. The author shows that a sufficiently exact solution can be obtained directly from a differential equation of Euler's variation problem without employing Ritz' method, provided the problem is confined to a consideration "in small." The problem is re-

duced to the solution of the equation

$$\frac{d^6v}{d\phi^6}+2\frac{d^4v}{d\phi^4}+(1+2k\cos^2\phi)\frac{d^2v}{d\phi^2}-2k\sin 2\phi\frac{dv}{d\phi}$$
$$-2k(2\sin^2\phi-\cos^2\phi)v=-k(3\sin 2\phi+(4/\pi)\epsilon\sin\phi),$$

where k, ϵ are constants, which solution satisfies the given boundary conditions on the contour of a cross-section of the shell. For a closed uncut shell the solution obtained is comparatively simple and is carried to completion.

Author's summary.

Teofilato, Pietro. Gli effetti del secondo ordine nelle vibrazioni elastiche. I and II. Pont. Acad. Sci. Acta **3**, 85–112 (1939). [MF 4098, 4099]

The purpose of I is to generalize the differential equations of elastic vibration by including terms quadratic in the derivatives of the displacement. The author assumes a stress-strain relation of the form $p_{ij}=(M-N)e_{ij}+Ne_{kk}\delta_{ij}$, where p_{ij} are the components of stress, M and N are elastic constants and e_{ij} are the usual expressions for finite strain components in terms of the displacement u_i and Cartesian coordinates x_i:

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\frac{\partial u_s}{\partial x_i}\frac{\partial u_s}{\partial x_j}.$$

It is not however made clear whether p_{ij} is the stress at x_i or at x_i+u_i, that is, whether the author is using the Eulerian or the Lagrangian method. Omitting details, the author uses an energy function $L=\frac{1}{2}N(e_{ii})^2+\frac{1}{2}(M-N)e_{ij}e_{ij}$ and Hamilton's principle to obtain equations of motion of the form $-\rho\partial^2u_i/\partial t^2+L_i=F_i$, where L_i is linear in the spatial derivatives of the displacement and F_i quadratic in these derivatives. In II the theory is applied to the vibrations of a tube of circular section subject to periodic pressures inside and out. Only a radial displacement is considered. The equation of motion is integrated by successive approximations, and the quadratic term is found to give rise to a second type of natural frequency.

It is not clear to the reviewer why the form chosen for the energy L does not include terms of the third order in e_{ij}; such terms would presumably give significant contributions to the equations of motion. The author does not refer to the work on finite strain by L. Brillouin [Les Tenseurs en Mécanique et en Élasticité, Paris, 1938] and F. D. Murnaghan [Amer. J. Math. **59**, 235 (1937)].

J. L. Synge (Toronto, Ont.).

Garcia, Godofredo. Lord Rayleigh's theory of seismic waves in a crystalline medium of the cubic system. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista **6**, 49–62 (1941). (Spanish) [MF 4605]

Public lecture.

Rosenblatt, Alfred. Sur la propagation des ondes de Rayleigh dans les milieux transversalement isotropiques (milieux de Rudzki). Revista Ci., Lima **42**, 901–916 (1940). [MF 4349]

Going back to the theoretical work of Rudzki and setting up an artificial medium, the author discusses very briefly the equations of motion which result from the elastic theory of small strains when there exists a strain-energy-function. He then substitutes the conditions for surface waves of the Rayleigh type and discusses the results in the form of inequalities.

J. B. Macelwane (St. Louis, Mo.).

MATHEMATICAL PHYSICS

Carathéodory, C. Das parabolische Spiegelteleskop. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 105–120 (1940). [MF 4409]

In a paper on the Schmidt telescope [Hamburg Math. Einzelschr. **28**, 36 pp. (1940); cf. these Rev. **2**, 138], the author developed a new idea in investigating higher order image errors. In this paper he applies his ideas to a parabolic mirror, comparing the results of his study with previous work of H. C. Plummer and K. Schwarzschild.

M. Herzberger (Rochester, N. Y.).

Epheser, Helmut. Eine moderne Darstellung der Gullstrandschen Arbeiten zur Strahlenoptik. Ann. Physik (5) **38**, 501–541 (1940). [MF 3896]

The paper derives anew the important results of Gullstrand using methods of tensor algebra. An introductory chapter gives the law of refraction, the integral invariants and the Lagrange brackets in a form valid for anisotropic and inhomogeneous media [compare M. Herzberger, Proc. Nat. Acad. Sci. U. S. A. **24**, 466–473 (1938)]. Use is made of Gullstrand's conceptions of optical projection and of Gullstrand's method of splitting up the four-dimensional manifold of light rays into two-dimensional manifolds determined either by the rays through a given object point or by the rays through a given point of the diaphragm.

Chapter one gives the equations for tracing a normal system of rays through an optical system. The connections are found between the Gaussian fundamental form of incident wave surface, refracting surface and emergent wave surface at the point of incidence. The law of Fermat gives the condition that the optical path between the two wave surfaces is constant. Differentiation leads to the refraction law; repeated differentiation gives the formulae of I. C. Sturm. Introduction of special coordinates leads to Gullstrand's equations. The second chapter repeats the investigation for inhomogeneous media. The formulae of Gullstrand are simplified by using Christoffel's symbolism. The third chapter studies the problem of imaging the whole four-dimensional manifold of neighboring rays. The author shows that Gullstrand's invariants can be interpreted as Lagrange brackets. Chapter four gives the fundamental equation of Gullstrand and the author's interpretation of it. He shows that every object surface contains two sets of curves to which correspond two sets of curves in the image space, not necessarily on the same image surface, such that all the rays passing through a point on one curve on the object surface must pass through the corresponding image curve.

The author makes no mention of the work of Herzberger who tried, in his book "Strahlenoptik" [J. Springer, Berlin, 1931], and in later publications, to achieve the same aim; namely, to present Gullstrand's fundamental ideas in modern form.

M. Herzberger (Rochester, N. Y.).

Graffi, Dario. Sull'applicazione del calcolo operatorio funzionale ai circuiti elettrici. Pont. Acad. Sci. Comment. **3**, 369–402 (1939). [MF 4102]

Pipes, Louis A. Transient analysis of symmetrical networks by the method of symmetrical components. Trans. Amer. Inst. Elec. Engrs. **59**, 457–459 (1940). [MF 3983]

The impedance matrix of an electrical network with n terminal pairs which is completely symmetrical, that is, invariant under the full symmetric group of pair permutations, contains only two independent impedance elements. It is well known that these are linear combinations of driving point impedances; this is seen from the equivalent diagonal form which is obtained by a symmetrical component transformation. This gives the description of both steady state and transient conditions in symmetrical networks. *H. G. Baerwald* (Cleveland, Ohio).

Pipes, Louis A. Transformation theory of general static polyphase networks. Trans. Amer. Inst. Elec. Engrs. **59**, 123–128 (1940). [MF 3982]

Application of matrix calculus to electrical power circuits. Typical three-phase circuits with general delta- and wye-connections are transformed into simple circuits. As a special application, the method of symmetrical components is represented from the transformation point of view. *H. G. Baerwald* (Cleveland, Ohio).

Amerio, Luigi. Tensioni e correnti in una catena di trasduttori quadripolari. Pont. Acad. Sci. Comment. **4**, 83–145 (1940). [MF 4109]

The steady state equations of recurrent quadripole cascades are derived in the usual way and specialized for the case of loaded lines, short circuited at the far end. The quadripole parameters of finite and infinite lines are rational and meromorphic functions, respectively, the real parts of whose singularities have a finite superior limit. Upon application of the Laplace transformation, the transient solutions for voltage and current in the nth member in terms of a general input signal are derived for the non-dissipative case for both finite and infinite lines. The solution is obtained in the two forms: in terms of eigen-oscillations (by partial fraction expansion) and in terms of progressive waves (power expansion), both derived in the usual way under application of involution. The asymptotic behavior of the single residue terms is discussed. Asymptotic (semiconvergent) expansions which, from the point of application, are an important form of the solution are not included. *H. G. Baerwald.*

Rice, S. O. Steady state solutions of transmission line equations. Bell System Tech. J. **20**, 131–178 (1941).

The steady state theory of uniform transmission lines is extended to the case of multiple lines (that is, consisting of several parallel wires). This generalization is essentially one of going over from relations of scalar variables to matrix relations. A square matrix Γ and its transposed, the square of the former being the product of the lengthwise-impedance matrix and crosswise-admittance matrix, are the generalizations of the propagation constant γ for a single circuit, and another square matrix Z_0 is that of the characteristic impedance. Matrix expressions involving Γ, Γ' and Z_0 are obtained for the voltages and currents. They are the analogues of the corresponding single current equations. The same methods are then applied to recurrent multi-terminal circuits consisting of a number of equal lumped symmetrical sections in tandem. Expressions for voltages and currents in circuits made up of unsymmetrical sections are also derived. They are naturally much more involved. The lines and sections may or may not contain distributed and lumped generators, respectively. The aspect of application, that is, of the practical manipulation of the matrix functionals, is stressed throughout. *H. G. Baerwald* (Cleveland, Ohio).

Grünberg, G. A. Eine Methode zur Lösung einer bestimmten Klasse elektrostatischer und verwandter Probleme. Acad. Sci. USSR. J. Phys. **3**, 401–416 (1940). [MF 3800]

The electrostatic problems treated in this paper consist in the determination of the electric field produced by a point charge in a region occupied by wedge-shaped dielectrics whose edges abut on a common axis, say, the z-axis. The method of solution utilizes Fourier integral expansion in z, whose coefficients are proper Bessel functions in r, the distance from the axis, and trigonometric in the azimuthal angle θ. The method of solution is applicable to diffraction problems. *H. Poritsky* (Schenectady, N. Y.).

Snow, Chester. Mutual inductance of two helices whose axes are parallel. J. Research Nat. Bur. Standards **25**, 619–671 (1940). [MF 3874]

Formulas are developed for the mutual inductance of non-coaxial finite helices with parallel axes. The currents are not closed and by "inductance" is meant the value of the Neumann integral extended over the helices. The inductance of proper helical current sheets constitutes the "principal part" of the inductance, but correction terms due to the finite number of turns are worked out in detail. The analytical work is very extensive and utilizes various Bessel function expansions (such as Neumann's and Sonine's expansions, Heine's integral formulas). At various stages cylindrical, spherical and oblate spherical coordinates are resorted to. Wherever possible a physical interpretation of the analytical steps has been given. *H. Poritsky.*

Hahn, W. C. A new method for the calculation of cavity resonators. J. Appl. Phys. **12**, 62–68 (1941). [MF 3664]

The method described consists of dividing the cavity into two or more regions, finding series expansions which satisfy the boundary conditions in each region separately, and matching the solutions at their common surface. It is applied only to cases in which the cavity is of such a shape that it can be divided into two simple regions by a cylindrical surface. The series obtained for these cases are slowly convergent; they are conveniently handled by subtracting them from certain standard series, thus obtaining rapidly convergent series which can be readily computed. Partial tables of these standard series are included. *R. M. Foster* (New York, N. Y.).

Arenberg, A. G. Les oscillations forcées d'un oscillateur sphérique dans un flux circulaire. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 147–150 (1940). [MF 3550]

The author presents a determination of the electromagnetic field exterior to a spherical oscillator, excited by an exterior magnetic field directed along the meridians. Expressions are derived for the equivalent magnetic dipole, the radiated energy and the radiation resistance. *R. M. Foster* (New York, N. Y.).

Jachnow, Walter. Über den Strahlungswiderstand eines geraden linearen Strahlers bei gedämpften fortschreitenden Wellen. Elektr. Nachr. Techn. **17**, 141–149 (1940). [MF 4437]

The radiation resistance R_s of a straight thin linear conductor is computed for the case that, by suitable feeder and

termination impedance, progressing attenuated waves are maintained on it. The mathematical procedure is straightforward, based on the Hertzian vector for the assumed current distribution. The integrals involved could not be evaluated in closed form. An approximation is obtained whose first term is identical with the radiation resistance for zero damping times $e^{-\beta l}$ (β=wave attenuation, l=length of radiator), and whose second term is of the order $(\beta l)^2$. An estimate of the latter is derived, which holds up to $\beta l \approx 0.5$. The dependence of βl on l/λ (λ=wavelength) is derived for the ideal case of the absence of losses other than due to radiation. Plots of R_s vs. l/λ from 0 to 10 for different βl from 0 to 0.5 and for two prescribed termination impedances ($Z=250$ and $500\ \Omega$) are given. *H. G. Baerwald.*

Smirnov, A. A. The problem of two plane waves in classical non-linear electrodynamics. Acad. Sci. USSR. J. Phys. **3**, 447–453 (1940). [MF 3799]

The problem of mutual interference of two plane monochromatic light waves is investigated by means of non-linear electrodynamics similar to the theory proposed by Born. It is shown that the perturbation method leads to a first order correction which does not remain finite at infinity. This is compared to what happens when the perturbation method is applied to non-linear vibration systems in mechanics when the first order terms lead to a forced oscillation in which the applied frequency is equal to the natural frequency of the system. It is pointed out that this might be remedied by expanding the frequency, too, in powers of the parameter. Applying an analogous method to the problem at hand, the author arrives at the following results for the effect of two waves passing through each other. The velocities of propagation of the initial fields are slightly changed and their electric and magnetic vectors differ slightly; there appear four small scattered waves with greatly different frequencies and velocities of propagation. *H. Poritsky* (Schenectady, N. Y.).

Rytow, S. M. und Judkewitsch, F. S. Über die Reflexion elektromagnetischer Wellen an einer Schicht mit negativer Dielektrizitätskonstante. Acad. Sci. USSR. J. Phys. **3**, 111–124 (1940). [MF 3728]

This paper continues the work of Hartree, Försterling, Epstein, Gans and others on the reflexion of electromagnetic waves by the Heaviside layer in the Jonosphere. The layer is represented by a region of variable dielectric constant which varies continuously in vertical direction beginning with $\epsilon=1$ and decreasing with increasing height down to $\epsilon=0$ and even to negative values. Gans had shown that for a small gradient of ϵ inside the layer there is no partial reflexion of the electromagnetic waves, but only total reflexion which occurs at those places where $\epsilon=0$. The authors confirm these results and show that for a greater gradient of ϵ there are regions of partial reflexion which change rapidly into regions of total reflexion at places where $\epsilon<0$. With oblique incidence there is even a stronger tendency towards total reflexion than with normal incidence.

The authors calculate for the first time the electric field near and inside the layer. As a consequence of the superposition of incident and reflected waves there is (especially for the case of total reflexion) a considerable reinforcement ("swelling") of the field at places adjoining the region $\epsilon<0$. It amounts to 5.4 times the original amplitude of the incident field if the height of the layer is 300 wavelengths. Therefore the electrons may receive an impulse from the electric field comparable to thermal motion. Their "electric" velocity may reach 20 percent of their thermal velocity. Consequently a considerable nonlinear distortion of modulated waves is caused and cross modulation between two such waves arises if the one is due to a near strong station influencing the layer and the other happens to pass through it (Luxemburg effect). In an appendix the density of electric energy is computed for cases of variable dielectric constant which may even assume negative values. *F. Kottler.*

Synge, J. L. On the electromagnetic two-body problem. Proc. Roy. Soc. London. Ser. A. **177**, 118–139 (1940). [MF 3757]

The author investigates the motion of two electrically charged bodies under the influence of their electromagnetic interaction according to classical electrodynamics. The following assumptions are made: (i) The particles are points with no volume; (ii) the electromagnetic field due to a moving charge is given by the usual retarded potential; (iii) a charged particle moves in accordance with the relativistic equation of motion based on the Lorentz ponderomotive force without a radiation damping term. The last assumption differs essentially from the one usually adopted and it invalidates the ordinary form of the conservation of energy. The author computes the rates of change of the mechanical energy and the angular momentum under the assumption that the ratio m'/m of the masses of the two particles is small. This is applied in detail to the case where the orbit of the light particle is approximately circular. The two particles spiral into an ultimate collision. If the velocity of the light particle is small compared with the velocity of light the energy decrease per unit time is $2m'/m$ times the rate which is usually assumed as a consequence of the radiation damping force. *W. Pauli* (Princeton, N. J.).

Molière, Gert. Zur Strahlungstheorie. I. Wellengleichung erster Ordnung für die Potentiale des Strahlungsfeldes. Ann. Physik (5) **37**, 415–420 (1940). [MF 4189]

It is shown that the Maxwell equations for the empty space become of first order if written in terms of four primary potentials which represent right and left circularly polarized progressing and regressing waves. This is of importance with respect to the application to the quantum theory of radiation, where first order equations are required for the potentials. It is shown that the Hamilton functions of the Schrödinger equations corresponding to the wave equations of the four above mentioned components can be brought into forms similar to the Dirac equation (but in three dimensions). The three-dimensional matrices involved fulfill corresponding cyclic permutation relations. No negative energy values are possible. *H. G. Baerwald.*

Whittaker, E. T. On Hamilton's principal function in quantum mechanics. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 1–19 (1941). [MF 3933]

In classical mechanics we have

$$\frac{\partial W}{\partial q}=p;\quad \frac{\partial W}{\partial Q}=-P;\quad \frac{\partial W}{\partial t}=-H.$$

q, p and Q, P are coordinates and momenta in the instants t and T, respectively. H is the Hamiltonian and W is the principal function defined by $W=\int_T{}^t L dt$ with L as the Lagrangian. The author analyzes the analogous situation in quantum-mechanics for a problem characterized by a given Hamiltonian $\overleftarrow{H}(\overleftarrow{q}, \overrightarrow{p})$ (the arrows indicate that $(\overleftarrow{q}, \overrightarrow{p})$ are non-commuting operators). It is shown that a function

$\tilde{U}$ exists ($\tilde{U}=\tilde{U}(\tilde{q},\tilde{Q},t-T)$) formally satisfying the same equations as W in classical mechanics with the exception of $W=\int_T^t L dt$ for which this analogy breaks down. Besides the Hamiltonian H and this quantum-mechanical principal function $\tilde{U}$ the author introduces the third principal function R gained from well-ordered U by replacing operators $\tilde{q}$, $\tilde{Q}$ by algebraic quantities q and Q. It is shown that $S(q,Q,t-T)=e^{i/\hbar R(q,Q,t-T)}$ satisfies Schrödinger's differential equation for the wave function belonging to the Hamiltonian $H(q,p)$. This theory enables the author to obtain Mehler's formula for Hermite's functions and Lebedeff's formula for Laguerre's functions, the first through the harmonic oscillator, the second through the Kepler problem.

L. Infeld (Toronto, Ont.).

Sommerfeld, A. und Hartmann, H. Künstliche Grenzbedingungen in der Wellenmechanik. Der beschränkte Rotator. Ann. Physik (5) **37**, 333–343 (1940). [MF 4188]

Let a two atom molecule have moments of inertia $J=J_1=J_2$, $J_3=0$. The well-known solution of the Schrödinger wave equation for the rotation is then (1) $\psi(\theta,\phi)=e^{\pm im\phi}P(x)$, where $m=0,1,2$ and $x=\cos\theta$. Write $j(j+1)=2JE/\hbar^2$, with E the energy. The conditions imposed are (a) continuity over $0\leqq\varphi<2\pi$, $\eta-1\leqq x\leqq 1$, (b) $P(1)=0$ and (c) $P(\eta-1)=0$. For fixed m and arbitrary j, under (a) and (b),

$$P(x)\equiv P_j^m(x)=(1-x^2)^{m/2}d^m/dx^m P_j(x),$$

where $P_j(x)$ is a hypergeometric function regular at $x=1$ but not, in general, at $x=-1$. Characteristic values for j, written j', are now determined by (c). (If $\eta=0$, the classic rotator case, $P_j(x)$ reduces to a Legendre polynomial and $j'=n$, $n=0,1,\cdots$.) Qualitative reasoning indicates the main features of the j' vs. η curves. The asymptotic approach of the j''s to the classic rotator values is treated for $m=0$ and $\eta\rightarrow 0$. The point of the analysis lies in circumventing the lack of regularity of $P_j(x)$ at $x=-1$ for j non-integral. Thus for instance $dj'/d\eta\rightarrow\infty$ as $\eta\rightarrow 0$ and

$$j'\sim n-1/(\log\eta/2+2(1+1/2+\cdots+1/n)),\quad n=1,2,\cdots.$$

D. G. Bourgin (Urbana, Ill.).

Belinfante, F. J. On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields. Physica **7**, 449–474 (1940). [MF 4362]

The author first investigates which conditions the Lagrangian function for an arbitrary field theory of matter has to fulfill in order that expressions for electric and current densities can be defined. It is shown that the general energy momentum tensor can be obtained from the Lagrangian function by introducing the condition that it should determine the gravitational field in a general relativistic formulation. The expressions for this tensor can, however, be obtained from the Lagrangian without reference to general relativity. Also suitable definitions for angular and spin momenta can be given with the help of the Lagrangian. In the second part the author investigates the ambiguities of the expressions for charge and momenta due to an addition of a divergence to the Lagrangian integrand. For a Lagrange function of first order (leading to linear field equations like the Maxwell equations) charge and current are uniquely determined. The interaction terms with the electromagnetic field follow then without ambiguity from the Lagrangian of the material part alone. This is, however, not true for a second order Lagrangian (leading to second order wave equations, as in electrodynamic potentials). It is further shown that the total energy and total angular momenta are always uniquely determined, their densities, however, in case of a second order Lagrangian, only if the latter is known in its full general relativistic form. Even then the separation of the spin and orbital parts of the total angular momentum remain indefinite. The densities of spin and angular momenta are indefinite in all cases also for a first order Lagrangian.

L. W. Nordheim (Durham, N. C.).

Bopp, Fritz. Eine lineare Theorie des Elektrons. Ann. Physik (5) **38**, 345–384 (1940). [MF 3782]

A classical theory of electrodynamics is developed in which a Maxwell field and a (real) Proca-Yukawa type field are superposed. The latter is interpreted as a polarization field similarly as in the theories of Mie and Born. The two fields are coupled only at singularities which are assumed to be sources for both fields and which are interpreted as electrons. If the signs of the two respective energy-momentum tensors are assumed to be opposite the infinities at the singularities cancel, and a singular solution exists with the finite energy $(1/2)\kappa e^2$, where e is the arbitrary charge and $(1/\kappa)$ the fundamental length in the Yukawa field. The mass of the quanta associated with the Yukawa field ($\mu=\hbar\kappa/c$) would thus be $2\hbar c/e^2=274$ times the rest mass of the electron $(1/2)\kappa e^2$. The interaction energy between two static charges is, corresponding to the assumed superposition, $\varphi=e_1e_2(1/r)(1-e^{-\kappa r})$. The field equations can be put into the Lagrangian form when derivatives of the field tensors are admitted in the Lagrange function. They can also be put into the Hamiltonian form. Approximate equations of motion can be obtained for the singularities by calculating the fields of the moving charges, neglecting their acceleration. The Lagrangian thus obtained coincides up to fourth powers in a development with respect to (v/c) with the expression of Darwin [Philos. Mag. **39**, 537 (1920)]. For the treatment of radiative reaction it is suggested that it should be derived from the postulate that energy and momentum flux through infinitesimal shells around the singularities should be zero, that is, that energy and momentum should be entirely contained in the field. This requires that the sum of the Lorentz forces of the inner and external field should vanish at the singularities. It is made plausible that this postulate is compatible with Dirac's theory of radiative reaction [Proc. Roy. Soc. London. Ser. A. **167**, 148 (1938)]. *L. W. Nordheim* (Durham, N. C.).

Fuchs, Klaus. Reciprocity. Part V: Reciprocal spinor functions. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 26–36 (1941). [MF 3934]

The scalar product of two spinor functions ψ_1, ψ_2 in r, t-space is defined so that it is invariant for rotations. It is also invariant for space-like reflections but changes sign for time-like reflections. With the aid of this scalar product the Fourier operator giving the invariant Fourier transformation of a spinor wave function is defined save for a normalizing factor which is taken to be unity for the Fourier transformation of the solutions of Dirac's wave equation. The reciprocal of this transformation is found and also a method of forming reciprocal spinor functions with the aid of the roots of the transcendental equation

$$2\pi\mu\{J^2_{k+\frac{1}{2}}(\mu)+J^2_{k+\frac{3}{2}}(\mu)\}=1.$$

The roots of this equation are given to two decimal places for values of k increasing by unit steps from $-\frac{1}{2}$ to $6\frac{1}{2}$.

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AUTHOR INDEX

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